THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and twenty-third regular meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City, on Saturday, April 29, 1905. The meeting extended through the usual morning and afternoon sessions. The attendance included the following thirty-eight members of the Society:

Professor D. P. Bartlett, Professor E. W. Brown, Professor F. N. Cole, Miss E. B. Cowley, Miss L. D. Cummings, Dr. D. R. Curtiss, Dr. W. S. Dennett, Dr. L. P. Eisenhart, Dr. William Findlay, Professor H. B. Fine, Dr. A. S. Gale, Professor L. I. Hewes, Mr. E. A. Hook, Mr. S. A. Joffe, Dr. Edward Kasner, Dr. O. D. Kellogg, Mr. E. H. Koch, Dr. G. H. Ling, Mr. L. L. Locke, Professor E. O. Lovett, Mr. R. B. McClenon, Professor H. P. Manning, Dr. Max Mason, Mr. J. C. Morehead, Dr. L. I. Neikirk, Professor W. F. Osgood, Professor James Pierpont, Miss Virginia Ragsdale, Mr. R. G. D. Richardson, Miss I. M. Schottenfels, Dr. Arthur Schultze, Professor Charlotte A. Scott, Professor D. E. Smith, Professor P. F. Smith, Professor H. D. Thompson, Miss M. E. Trueblood, Professor E. J. Wilczynski, Mr. J. E. Wright.

The President of the Society, Professor W. F. Osgood, occupied the chair, being relieved by Vice-President E. W. Brown and the Secretary. The Council announced the election of the following persons to membership in the Society: Mr. J. H. Grace, Peterhouse, Cambridge, England; Mr. H. B. Leonard, University of Chicago; Mr. R. B. McClenon, Yale University; Mr. W. S. Monroe, Columbia Normal Academy, Columbia, Mo.; Mr. J. C. Morehead, Yale University; Professor Henri Poincaré, University of Paris; Mr. R. G. D. Richardson, Yale University; Miss S. F. Richardson, Vassar College; Mr. F. R. Sharpe, Cornell University; Miss M. S. Walker, University of Missouri. Six applications for membership in the Society were received.

The following papers were read at this meeting:

(1) Dr. ARTHUR SCHULTZE: "Graphic solution of quadratics, cubics and biquadratics."

(2) Dr. MAX MASON: "On the derivation of the differential equation of the calculus of variations."

(3) Dr. D. R. CURTISS: "Theorems converse to Riemann's on linear differential equations."

(4) Miss VIRGINIA RAGSDALE: "On the arrangement of the real branches of plane algebraic curves."

(5) Mr. J. C. MOREHEAD: "Numbers of the form $2^k q + 1$ and Fermat's numbers."

(6) Professor E. B. VAN VLECK: "Some theorems of pointwise discontinuous functions; supplementary note."

(7) Professor JAMES PIERPONT: "Inversion of double infinite integrals."

(8) Professor JAMES PIERPONT: "Multiple integrals (second paper)."

(9) Mr. R. B. MCCLENON: "On simple integrals with variable limits."

(10) Professor E. O. LOVETT: "On a problem including that of several bodies and admitting of an additional integral."

(11) Professor M. B. PORTER: "Concerning Green's theorem and the Cauchy-Riemann differential equations."

(12) Professor M. B. PORTER: "Concerning series of analytic functions."

(13) Mr. J. E. WRIGHT: "Differential invariants of space."

(14) Dr. EDWARD KASNER: "On the trajectories produced by an arbitrary field of force."

(15) Dr. E. B. WILSON: "Sur le groupe qui laisse invariant l'aire gauche."

(16) Professor E. J. WILCZYNSKI: "Projective differential geometry."

(17) Miss I. M. SCHOTTENFELS: "On the simple groups of order $8!/2$" (preliminary communication).

(18) Miss I. M. SCHOTTENFELS: "Certain trigonometric formulæ for the quantity $x + \epsilon y$, where $\epsilon^2 = 0$."

(19) Dr. EDWARD KASNER: "A theorem concerning partial derivatives of the second order, with applications."

(20) Mr. J. E. WRIGHT: "On differential invariants."

(21) Dr. L. P. EISENHART: "Surfaces of constant curvature and their transformations."

(22) Professor L. E. DICKSON: "On the class of the substitutions of various linear groups."

(23) Professor JOSIAH ROYCE: "The fundamental relations of logical and geometrical theory."

Professor Royce's paper was communicated to the Society through Professor Moore. In the absence of the authors, Professor Porter's papers were presented by Dr. Curtiss and Professor Pierpont, and Dr. Wilson's paper by Dr. Gale. The papers of Professor Lovett, Dr. Eisenhart, Professor Dickson, Professor Royce, Dr. Kasner's second paper and Mr. Wright's second paper were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The usual method of solving equations graphically requires the plotting of a curve for each individual equation — a method too cumbersome to be of great practical value. Dr. Schultze's paper shows that there are certain classes of equations which can be solved by the construction of one curve for each class, and the use of ruler and compasses. The method employed consists in reducing an equation involving one unknown quantity to a system of two simultaneous equations. One of these equations is identical for all equations of the same type, while the other is the equation of a straight line or a circle. By this method all quadratics can be solved graphically by means of the parabola $y = x^2$ and straight lines, or the hyperbola $y = 1/x$ and straight lines; cubics by the cubic parabola $y = x^3$ and straight lines; biquadratics by a parabola and circles. While the main application of the method consists in solving equations of the fourth and lower degrees, it can be employed for certain other equations, for example, Kepler's equation $x - \epsilon \sin x = c$.

2. In Dr. Mason's paper the differential equation which arises from the vanishing of the first variation of multiple integrals was derived by a method analogous to that used by Du Bois Reymond in the case of simple integrals, without assuming the existence of derivatives of the dependent variable higher than those which occur in the integral.

3. In the unfinished paper "Zwei allgemeine Sätze" etc., numbered XXI in his collected works, Riemann develops the following two properties of $n + 1$ linear families made up of the solutions of $n + 1$ homogeneous linear differential equations of the nth order having the same group: 1) All n-rowed determinants of corresponding bases of these families undergo the same substitutions, which are merely multiplicative. 2) There exists a linear relation with single-valued coefficients be-

tween corresponding branches. In Dr. Curtiss's paper theorems converse to these are investigated, criteria thus being obtained that systems of families should have the same group. These results are then illustrated by applications to equations of the second order.

4. The only processes by which curves with the maximum number of circuits have been obtained are the two processes employed by Harnack and Hilbert. Miss Ragsdale's paper shows that on non-singular curves of even order derived by these two processes the circuits conform to a noteworthy law of arrangement. If an *internal* oval of a curve $u = 0$ be defined as an oval which cuts off in the midst of a region where u is negative a region where u is positive, and if an *external* oval be similarly defined, this law can be stated as follows: The number of internal ovals on a non-singular $2n$-ic can not be less than $\frac{1}{2}(n-1)(n-2)$ if the maximum number of circuits is present; and the number of external ovals can not exceed $n^2 + \frac{1}{2}(n-1)(n-2)$ whatever the number of circuits.

Though the proof given applies only to curves derived by these processes, it appears almost certain that the law, as stated, is perfectly general.

5. The first part of Mr. Morehead's paper gives a method and a description of a table for determining expeditiously the primes and the prime factors of the non-primes of the form $2^k q + 1$, k and q being positive integers. The method and the formation of the table are based on several theorems on the form q must have in order that $2^k q + 1 \equiv 0$, mod p, p being an odd prime. With a slight modification these theorems apply equally to the form $2^k q - 1$.

The second part of the paper applies these results to the investigation of possible factors of Fermat's numbers $F_n = 2^{2^n} + 1$, which factors must have the form $2^{n+2} \cdot q + 1$. The net result is that, in addition to the known factors of F_5, F_6, F_9, F_{11}, F_{12}, F_{18}, F_{23}, F_{36}, F_{38}, no more factors under 524,288,000 exist for F_n, $n > 16$, as against the limit 100,000,000 given by Cunningham and Western in the *Proceedings of the London Mathematical Society*, series 2, volume 1 (1904), page 175.

6. Professor Van Vleck's note, supplementary to his October paper on pointwise discontinuous functions, consisted of an

announcement of the two following theorems: 1) If $f(x, y)$ is continuous with respect to x when y is constant, and with respect to y when x is constant, in a field bounded by $x = a$, $x = b$, $y = c$, $y = d$, then

$$F(y) \equiv \int_a^b f(x, y)dx$$

is at most a pointwise discontinuous function for $c \leqq y \leqq d$. 2) If, furthermore, $\partial F/\partial y$ exists in the same field and is continuous with respect to x and y separately, then there is a set of points everywhere dense in (c, d) at which $F(y)$ has a derivative and at which, moreover, the value of $F'(y)$ can be obtained by differentiation of $f(x, y)$ with respect to y under the integral sign.

7. By extending results obtained by Vallée-Poussin in the prize memoir (Brussels Academy, volume 16, 1891–2), Professor Pierpont arrived at the following theorem whose generalization he reserved for another occasion:

Let $f(x, y)$ have no points of infinite discontinuity except on a finite number of lines parallel to the X and Y axes. Let

$$\int_a^\infty f(x, y)dx, \qquad \int_a^\infty f(x, y)dy$$

be in general uniformly convergent in any finite intervals (a, b), (a, β) respectively. Let either

$$\int_a^\infty dy \int_a^x f\,dx \quad \text{or} \quad \int_a^\infty dx \int_a^y f\,dy$$

be uniformly convergent in (a, ∞), (α, ∞) respectively. Then both

$$\int_a^\infty dy \int_a^\infty f\,dx, \qquad \int_a^\infty dx \int_a^\infty f\,dy$$

exist and are equal.

8. Professor Pierpont's second paper related to the reduction of m-tuple proper integrals to iterated integrals, also to the change of variables in such integrals. The only *general* and rigorous treatment of these questions is that of Jordan, for the case of $m = 2$. His methods do not seem to admit of simple general-

ization for higher cases. By an entirely different principle Professor Pierpont establishes the fundamental relation of the type

$$\underline{\int}_{\mathfrak{A}} f\,dx_1\cdots dx_m \leq \underline{\int}_{\mathfrak{r}_m} dx_m \underline{\int}_{\mathfrak{B}_m} f\,dx_1\cdots dx_{m-1}$$

$$\leq \overline{\int}_{\mathfrak{r}_m} dx_m \underline{\int}_{\mathfrak{B}_m} f\,dx_1\cdots dx_{m-1} \leq \overline{\int}_{\mathfrak{A}} f\,dx_1\cdots dx_m$$

and the relation

$$\underline{\int}_{\mathfrak{A}} f\,dx_1\cdots dx_m = \overline{\int}_{\mathfrak{B}} f\,\frac{\partial(x_1\cdots x_m)}{\partial(u_1\cdots u_m)}\,du_1\cdots du_m,$$

on the assumption that f is limited in the field $\mathfrak{A}$, concerning whose boundary no restrictions are made except that it is discrete.

9. In Mr. McClenon's paper functions defined by the definite integral

$$J(y) = \int_{\phi(y)}^{\psi(y)} f(x, y)dx$$

are considered. Various conditions for the continuity and differentiability of such functions are established; and in particular it is proved that the formula

$$(1)\quad \frac{dJ}{dy} = \int_{\phi(y)}^{\psi(y)} f'_y(x, y)dx + f[\psi(y), y]\psi'(y) - f[\phi(y), y]\phi'(y),$$

$\phi(y)$ and $\psi(y)$ being one-valued and differentiable, holds true when $f(x, y)$ and $f'_y(x, y)$ have discontinuities along a finite number of lines parallel to the Y axis, provided the singular integrals for those lines converge uniformly to zero for every y in the interval in question. In the case

$$J(y) = \int_{\phi(y)}^{\infty} f(x, y)dx,$$

if (1) fails to define a function, owing to the non-convergence of the integral

$$\int_{\phi(y)}^{\infty} f'_y(x, y)dx,$$

$J(y)$ may still be differentiable, under conditions analogous to those of Vallée-Poussin for constant limits, viz.,

$$\int_{\phi(y)}^{X} f_y'(x, y)dx = g(X, y) + \int_{\phi(y)}^{X} h(x, y)dx$$

where $\int_{\phi(y)}^{\infty} h(x, y)dx$ is uniformly convergent and

$$\int_b^y g(X, y)dy = 0$$

as X increases without limit. In fact, under these assumptions,

$$\frac{dJ}{dy} = \int_{\phi(y)}^{\infty} h(x, y)dx - f[\phi(y), y]\phi'(y).$$

The result is that a number of the most important theorems concerning integrals with fixed limits of integration are shown to be valid in the more general case of variable end points.

10. In the problem of three bodies Bertrand introduced certain quadratic functions of the coördinates of the bodies and of quantities proportional to the projections of the velocities on the axes of coördinates. Bour showed that Bertrand's variables satisfy a certain system S of ordinary differential equations of the first order, and pointed out that the problem of three bodies may be considered as a particular solution of a more general problem whose equations are S and of which a certain integral D is known. It is the object of Professor Lovett's note to carry out the extension of these results to the case of any number of bodies. The paper will appear in the *Transactions*.

11. Professor Porter's first paper gave an elementary and simple proof of the theorem

$$\int_c f\,dy = \int_R\int f_x'\,dxdy,$$

if 1) $f(x)$ is continuous in T; 2) f_x' is limited and integrable in T, in Riemann's sense; 3) f_x' exist in T except perhaps for a point set of (two dimensional) content zero. The case when f_x' becomes infinite in T was also considered, and it was pointed out that if

a pair of functions u and v satisfying the Cauchy-Riemann differential equations be such that their first partial derivatives with respect to x and y satisfy conditions 2) and 3), $u + vi$ will be analytic in T, so that it is not necessary to suppose that $f'(z) = (u + iv)'$ exist throughout T in demonstrating Cauchy's integral formula.

12. Professor Porter's second paper was concerned with the proof of the following theorem: If

$$\sum_1^n f_n(z) \qquad (n = 1, 2, \cdots),$$

where $f_i(z)$ is analytic in a given region γ, is *limited* in γ, then

$$\lim_{n=\infty} \sum_1^n f_n(z)$$

will define a set of functions analytic in γ whose number is at least as great as the number of limiting points of

$$\sum_1^n f_n(z_0) \qquad (n = 1, 2, \cdots),$$

where z_0 lies in γ. From this theorem was deduced the conclusion that, if

$$\lim_{n=\infty} \sum_1^n f_n(z)$$

converged for a point set one at least of whose limiting points is in γ, then

$$\sum_1^\infty f_n(z)$$

converges throughout any region of γ to a unique analytic function. This paper will appear in the *Annals of Mathematics*.

13. In Mr. Wright's first paper a functionally independent complete system of differential invariants is obtained as the set of algebraic invariants of a system of m-ary forms, where the space is of m dimensions. Those which appertain to a given manifold of dimensions r are next obtained as the algebraic invariants of a system of r-ary forms. From these are deduced the general forms which determine the differential parameters of a

quadratic differential form. By the use of another method there is obtained a complete system of invariants and parameters of any order for any number of differential forms of any degrees whatever.

14. The equations of motion of a particle in a plane acted upon by a force depending only on position are of the form $\ddot{x} = \phi(x, y)$, $\ddot{y} = \psi(x, y)$, where the dots indicate differentiation with respect to the time. There are ∞^3 possible trajectories, whose differential equation is of the type $y''' = Py'' + Qy''^2$, where P and Q involve x, y, y'. Dr. Kasner discusses the properties of such triply infinite systems of curves. For a given initial speed v_0, there are ∞^1 trajectories through each point. Their centers of curvature are situated on a straight line. There thus arises a correspondence between the points and the straight lines of the plane. If the correspondence is given (arbitrarily), the force is completely defined except in intensity. When the field is lamellar, that is, when the force is conservative, the trajectories are related to the conformal representation of geodesics; when the field is solenoidal, they are related to isogonal trajectories. The case where the correspondence reduces to a reciprocal transformation is discussed in detail.

In the case of central forces, the trajectories fall into ∞^1 doubly infinite systems, one for each value of the sectorial velocity. The only cases where these systems are linear arise when the law of force is of the form $ar^{-2} + br^{-3}$. In the general case, the trajectories through a given point corresponding to a given sectorial velocity have their centers of curvature on a rational cubic curve.

The properties stated are essentially kinematic, since they involve the idea of velocity. Among the purely geometric results, the following is noteworthy. Through any period there pass ∞^1 trajectories in an assigned direction; each of these determines a unique osculating parabola; the foci of these parabolas lie on a circle. Every field of force thus gives rise to a correspondence between the triply infinite manifolds, lineal elements and circles.

In conclusion two general problems of interest were indicated: First, the determination of all forces whose trajectories constitute a linear system; second, the determination of all forces whose trajectories admit continuous groups. The familiar central forces (elastic law and Newton's law) possess both of these properties.

15. In the *Jahresbericht der deutschen Mathematiker-Vereinigung*, volume 12 (1903), Dr. Wilson gave a generalized definition of area and volume independent of the conception of length and especially suitable for use in projective geometry. This was applied to a discussion of the projective group in the plane with particular reference to transformations resoluble into two projective reflections. See *Annals of Mathematics*, volume 5 (1903). M. Fréchet has generalized the notion of area so as to obtain the area of a twisted curve in space. See *Nouvelles Annales de Mathématiques*, June, 1904. In the present note Dr. Wilson determines the group that leaves area invariant in space. This group is identical with that which leaves length invariant. The analysis is carried out with particular ease by using the methods introduced into multiple algebra by J. Willard Gibbs.

16. Professor Wilczynski showed that projective differential geometry is closely bound up with the theory of invariants of linear differential equations, or of systems of such equations, ordinary or partial. He gave a brief outline of the theory of plane and space curves, and of ruled surfaces as developed by him under the auspices of the Carnegie Institution. A systematic account of this theory will be found in a volume which is soon to appear, published by Teubner. He also indicated how a general theory of surfaces may be constructed, governed by the same general ideas.

17. In Miss Schottenfels's first paper, it is shown that all simple groups of order 20160 can be expressed as substitution groups upon 35 or a smaller number of letters.

18. Miss Schottenfels's second paper is devoted to a development of the analogues of the ordinary trigonometric formulas for the hypercomplex variable $x + \epsilon y$ where $\epsilon^2 = 0$.

19. The theorem of Dr. Kasner's second paper concerns the condition which must be satisfied by three functions A, B, C of two variables x, y if it is possible to find a function f so that $A : B : C = f_{xx} : f_{xy} : f_{yy}$. Application is made to the inverse problem of the calculus of variations, to a certain class of point transformations, and to the asymptotic lines of surfaces. For example, it is shown that when the asymptotic lines are projected orthogonally upon a plane, the result is an isothermal net whenever it is an orthogonal net.

21. Dr. Eisenhart considers a surface of constant positive curvature referred to an isothermal-conjugate system of lines. When the parameters are suitably chosen, the two fundamental quadratic forms and the equations of condition defining their coefficients can be so written as to disclose the existence of a second surface with the same second quadratic form and total curvature whose linear element is obtained by interchanging the first and third coefficients of the linear element of the given surface and changing the sign of the second coefficient. When the given isothermal-conjugate system is composed of the lines of curvature, which is possible for every surface of constant curvature, this transformation is the one first considered by Hatzidakis and called by Bianchi the Hatzidakis transformation. It is shown that the transformation for any isothermal-conjugate system gives the same surface as the Hatzidakis transformation applied to the lines of curvature. In the foregoing transformation the lines of curvature correspond.

Another transformation is found which enables one to derive from a given surface with constant positive curvature a second surface with the same total curvature and second quadratic form, the lines of curvature on the one surface corresponding to the characteristic conjugate lines on the other. The foregoing results are applied to surfaces of revolution with constant positive curvature.

22. Professor Dickson's paper appeared in the May number of the BULLETIN.

23. The paper of Professor Royce is devoted to a restatement and development of the theory of "the relation between the logical theory of classes and geometrical theory of points," which Mr. A. B. Kempe outlined in the *Proceedings of the London Mathematical Society* in 1890. As Mr. Kempe left the statement of the theory in question, it was an account of a development of the principles of projective geometry as a special instance of the laws which are at the basis of the algebra of logic. In the present restatement, the principles of the algebra of logic themselves receive a new formulation differing both from Kempe's and from the various systems of postulates for the algebra of logic recently expounded by Dr. Huntington, in the *Transactions*. A fundamental relation, the "O-relation," is defined by postulates. A system of entities that may or may not stand to one another in O-relations is described.

The O-relations are wholly symmetric, and may hold for pairs, triads, or in fact for any multitude of the elements. The unsymmetric relations possible in the algebra of logic are deduced from the properties of the O-relations. A new theory of the nature of logical addition and multiplication is developed, founded on Kempe's, but involving a considerable generalization of his procedure. It is then shown how the axioms of geometry can be proved as theorems of the algebra of logic, when the entities to which these theorems are to be applied are chosen in a special way. F. N. COLE,

Secretary.

THE APRIL MEETING OF THE CHICAGO SECTION.

THE seventeenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago on Saturday, April 22, 1905. The total attendance was forty-five, including the following members of the Society:

Professor G. A. Bliss, Professor Oskar Bolza, Dr. W. H. Bussey, Professor E. W. Davis, Professor L. E. Dickson, Dr. Saul Epsteen, Mr. M. E. Graber, Professor G. W. Greenwood, Professor E. R. Hedrick, Professor T. F. Holgate, Mr. A. E. Joslyn, Dr. H. G. Keppel, Professor Kurt Laves, Mr. N. J. Lennes, Dr. A. C. Lunn, Mr. J. H. Maclagan-Wedderburn, Professor Heinrich Maschke, Professor E. H. Moore, Professor F. R. Moulton, Professor H. L. Rietz, Professor J. B. Shaw, Professor H. E. Slaught, Dr. W. M. Strong, Professor H. S. White, Mr. N. R. Wilson, Dr. J. W. Young, Professor J. W. A. Young.

The first session was called to order at ten o'clock. Professor E. R. Hedrick was elected chairman, and Dr. J. W. Young secretary pro tem. On motion of Professor Moore it was voted that the chair appoint a committee to consider means of improving the meetings of the Section. The chair appointed on this committee Professors Dickson, Davis and Bliss.

The following papers were read:

(1) Dr. SAUL EPSTEEN and Mr. R. L. MOORE: "A set

of postulates for the theory of matrices (preliminary communication)."

(2) Mr. R. L. MOORE: "Sets of metrical hypotheses for geometry."

(3) Professor J. B. SHAW: "Groups of quaternions."

(4) Professor J. B. SHAW: "General algebra."

(5) Professor H. L. RIETZ: "Simply transitive primitive groups which are simple groups."

(6) Mr. N. R. WILSON: "Simple groups of even order less than 5000."

(7) Professor L. E. DICKSON: "On the cyclotomic function."

(8) Professor ALFRED LOEWY: "Ueber die vollständig reduciblen Gruppen, die in eine Gruppe linearer homogener Substitutionen gehören."

(9) Professor HEINRICH MASCHKE: "Differential parameters of the first order."

(10) Professor HEINRICH MASCHKE: "The Gaussian curvature of hyperspace."

(11) Professor G. A. BLISS: "A theorem of Volterra concerning line functions."

(12) Professor G. A. BLISS: "The inverse problem in the calculus of variations in parametric representation."

(13) Mr. A. L. UNDERHILL: "The invariantive property of the four necessary and sufficient conditions of the calculus of variations under contact transformations (preliminary communication)."

(14) Mr. M. E. GRABER: "On the definition and classification of cyclidal surfaces."

(15) Professor L. E. DICKSON: "On the equations of geometry."

(16) Dr. OSWALD VEBLEN: "Projective geometry in an arbitrary field."

(17) Dr. OSWALD VEBLEN and Dr. W. H. BUSSEY: "Finite projective geometries."

(18) Mr. J. H. MACLAGAN-WEDDERBURN: "A theorem on finite algebras."

(19) Mr. F. W. OWENS: "The Desargues theorem and the ideal elements in projective geometry (preliminary communication)."

(20) Dr. OSWALD VEBLEN and Mr. J. H. MACLAGAN-WEDDERBURN: "Non-desarguesian and non-pascalian geometries."

(21) Dr. OSWALD VEBLEN: "Note on the square root and the relations of order."

(22) Professor L. E. DICKSON: "On hypercomplex number systems."

Professor Loewy's paper was communicated to the Society through Professor Moore; Mr. R. L. Moore was introduced by Dr. Epsteen, and Mr. Owens by Dr. Veblen. In the absence of the author, Professor Loewy's paper was presented by Professor Dickson. The papers numbered 9, 16, 21, 22 were read by title. The meeting adjourned at a quarter past six o'clock.

Abstracts of the papers presented follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper the authors presuppose the existence of a field F with its usual properties. The undefined elements are vid, matrix, $\odot$, $\otimes$, $\oplus$. The postulates are nine in number and they are independent if the field F contains a mark which is not an integer. Otherwise one of the postulates is dependent upon the other eight which are themselves independent.

It is worthy of note that the associative and distributive laws of multiplication are not introduced by postulates. The former is a theorem, the latter is an easy consequence of the postulates and one of the definitions.

2. Mr. Moore's paper contains a set of independent postulates for euclidean geometry with point, order, and congruence (of segments) as undefined symbols. Congruence for angles is introduced by definition. If in place of a certain continuity postulate is substituted the postulate that every segment has a midpoint, there follows that portion of Hilbertian plane geometry which is not partly or wholly dependent on "the axiom of Archimedes." If instead of this midpoint postulate it is assumed that the sum of two sides of every triangle is greater than the third side and that every two right angles are equal, then Hilbertian three-dimensional geometry follows (with the same restriction as above). Other alternative sets of postulates are discussed and among other results is obtained the theorem that if to these postulates is added the assumption that the sum of the angles of every triangle is two right angles, then ideal points may be introduced in such a manner that the thus ex-

tended geometry is euclidean though (as has been shown by Dehn) the original might not have been.

3. In this paper the isomorphism between a quaternion and a linear fractional substitution $\begin{pmatrix}\alpha\beta\\ \gamma\delta\end{pmatrix}$ is made use of to develop the existence of finite groups of quaternions. These may be isomorphic to the rotation groups, extended rotation groups, congruence groups, and groups with α, β, γ, δ in any abstract field.*

4. The four deep principles of mathematics are taken to be the ensemble, characteristic of arithmetic; the correspondence, characteristic of the theory of functions; generation, characteristic of groups; relation, characteristic of algebras. Relations are of second, third, or higher rank, according as they are between two, three, or more elements. Relations of a given rank are uniform when, if $R_{12\ldots n}$ be the relation, of rank n, there are $n!$ relations existing: $R_{\alpha\beta\ldots\nu}$; where $\alpha, \beta, \ldots, \nu$ is any permutation of $1, 2, \ldots, n$, and all the relations exist for any given set of n elements $a, b, \ldots, n$. These relations combine in a manner isomorphic with the symmetric substitution group on n letters. These lead in the case of rank n to n sets of n identities corresponding to the combinations $R_{\alpha'\beta'\ldots\nu'}$ $R_{\alpha''\beta''\ldots\nu''}$. Other equations may be added to these, which are called limitations and furnish the defining limitations of particular types of algebras, as, *e. g.*, associative algebras. When different relations, as R', R'', $\ldots$, $R^{(m)}$, exist in an algebra, the algebra is of grade m corresponding. We thus have distributivity and other like relations.

5. So far as the determination of the primitive groups of a given degree has been carried out, there occur only two simply transitive primitive groups which are simple groups of composite order. It is the object of Professor Rietz's paper to call attention to a system of such groups. It is proved that there is a group of this type whose degree is congruent to 1 modulo p corresponding to every prime number $p > 11$, when p is itself twice a prime number plus one.

* Mr. Wedderburn has called my attention to the paper by W. I. Stringham. *American Journal*, vol. 4 (1882) pp. 345–358, on "Determination of the finite quaternion groups," where the same results are obtained. J. B. S.

6. The question of the existence of simple groups of even orders less than 2000 has been dealt with by Hölder, Cole, Burnside, and Ling and Miller. The object of Mr. Wilson's paper is to continue the search up to the order 5000. It is shown that between these limits, there can be no simple group of even order other than 2448, 2520, 3168, 3420, 3744 and 4080, and that in the first, fourth, and sixth cases such a group must be holoedrically isomorphic with the linear fractional groups of these orders, and in the second case with the alternating group of degree 7. The question of the existence of simple groups of the remaining two orders is not yet answered.

7. The first paper by Professor Dickson treats of the cyclotomic function $Q_n(x)$ whose roots are the primitive nth roots of unity. The usual derivation of the explicit formula for Q is essentially a verification; the present paper obtains Q by means of the recursion formula $Q_n(x) = Q_\nu(x^{p^r}) \div Q_\nu(x^{p^{r-1}})$, where $\nu = n/p^r$, p^r being the highest power of p which divides n. For x integral, a common divisor of $Q_n(x)$ and $x^m - 1$ $(m < n)$ must be the greatest prime p dividing n, and then x must belong to the exponent ν modulo p. Further, Q is not divisible by p^2 except in the trivial case $n = p = 2$, $Q = x + 1$, $x \equiv 3 \pmod 4$. Aside from this exception, and the case $n = 6$, $x = 2$, $Q_n(x)$ and hence $x^n - 1$ has a prime factor not dividing $x^m - 1$ $(m < n)$. The last result yields an immediate proof of the theorem that the commutativity of multiplication is a consequence of the other postulates for a finite field, the double distributive law included. The paper is to appear in the *American Mathematical Monthly* for April, 1905.

8. Professor Loewy calls a linear homogeneous group G completely reducible when, after a suitable linear transformation of variables, the new variables fall into systems such that G permutes amongst themselves the variables of each system according to an irreducible group. It is shown that every linear group which leaves invariant a positive Hermitian form is completely reducible. By a theorem published simultaneously by Loewy and Moore, every finite linear group possesses an invariant positive Hermitian form, and hence is completely reducible, a result due to Frobenius and Maschke. But an infinite linear group need not be completely reducible. Loewy shows that to every linear group G belong μ completely re-

ducible groups; these are uniquely determined in a definite sequence, if similar groups are not regarded as different. This theorem is in contrast to that on the irreducible Theilgruppen of G, which are uniquely determined apart from sequence. The theorems are extended to linear groups in infinite domains. The paper is to appear in the October number of the *Transactions*.

9. Differential parameters of the first order are given in the symbolic representation by

$$(U', \cdots, U^\lambda, f, \cdots, f^{n-\lambda})(V', \cdots, V^\lambda, f \cdots, f^{n-\lambda}),$$

where $U', \cdots, U^\lambda$ and $V', \cdots, V^\lambda$ are functions of $x_1, \cdots, x_n$ and $f, \cdots, f^{n-\lambda}$ equivalent symbols of the differential quadratic quantic $ds^2 = \sum_{i,k=i}^{n} a_{ik} dx_i dx_k$. Professor Maschke's first paper deals particularly with the case where these differential parameters vanish. For example the following theorems are demonstrated:

In order that every direction of the space S_λ: $U' = \text{const.}, \cdots, U^{n-\lambda} = \text{const.}$ may be orthogonal to every direction of the space $S_{n-\lambda}$: $V' = \text{const.}, \cdots, V^\lambda = \text{const.}$ at a common point, the equations must be satisfied

$$(V^i, f, \cdots, f^{n-1})(U^k, f', \cdots, f^{n-1}) = \sigma$$
$$(i = 1, \cdots, \lambda\,;\ k = 1, \cdots, n - \lambda).$$

The condition that each of the two spaces of λ dimensions

$$U' = \text{const.}, \cdots, U^{n-\lambda} = \text{const.};$$
$$V' = \text{const.}, \cdots, V^{n-\lambda} = \text{const.},$$

contain at a common point one direction which is orthogonal to all directions of the others, is

$$(U', \cdots, U^\lambda, f', \cdots, f^{n-\lambda})(V', \cdots, V^\lambda, f', \cdots, f^{n-\lambda}) = \sigma.$$

A number of theorems of a similar nature have been developed in the paper.

10. In this paper Professor Maschke deduces the explicit expression of an invariant of the general quadratic differential quantic $ds^2 = \sum_{i,k=1}^{n} a_{ik} dx_i dx_k$ which, in the case where the space

of n dimensions whose ds^2 is determined by the above quantic lies in an euclidean space of $n+1$ dimensions, coincides precisely with Kronecker's extension of the gaussian curvature. This invariant, which is given by two entirely different expressions for odd and even values of n, might then properly be called the gaussian curvature of the quadratic quantic ds^2.

This gaussian curvature is also developed for those spaces of λ dimensions which are defined by $n-\lambda$ equations

$$U' = \text{const.}, \cdots, U^{n-\lambda} = \text{const.}$$

Here the gaussian curvature appears as a differential parameter of the second order including the functions $U', \cdots, U^{n-\lambda}$.

11. Volterra was apparently the first to make a systematic study of functions whose arguments are arcs of curves. He derived the important theorem that the so-called first variation of the function is expressible in the form

$$\delta F = \epsilon \int_{x_0}^{x_1} F' \eta(x) dx,$$

where F is the function in question, F' its generalized derivative, and the argument and its variation have the form $y = \phi(x)$ and $y = \phi(x) + \epsilon\eta(x)$ respectively. This formula has many applications but loses much of its effectiveness if the argument is restricted to the form $y = \phi(x)$. The object of Professor Bliss's paper is to simplify the proof, and to extend the theorem to curves in parametric representation.

12. The inverse problem of the calculus of variations in the simplest case is: given a differential equation of the form

$$y'' = f(x, y, y'), \tag{1}$$

to find the integrals

$$I = \int g(x, y, y') dx,$$

whose extremals are defined by the equation (1). In this form the problem has been solved by Darboux and Hirsch, and the results have been applied by Hamel, among others, in his extremely interesting dissertation "Ueber die Geometrien in denen die Geraden die Kürzesten sind." A study of this paper impresses one, however, with the difficulties which are artificially introduced when the curves considered are restricted to the

form $y = \phi(x)$. In Professor Bliss's second paper this restriction is removed, and the results extended to curves in parametric representation.

13. The general problem in Mr. Underhill's paper is to show the invariance of the four necessary and sufficient conditions in the calculus of variations under contact transformation. The conditions are designated by E, L, J, W. The problem is of order n when n derivatives occur under integrand function. The invariance has been shown in the following cases: (*a*) Non-parametric form, (1) E, L, J, W, for extended point transformation, first and second order problems; (2) E, for Lie contact transformation, first order problem; (3) E, L, J, W, for Lie contact transformation, second order problem. (*b*) Parametric form, (1) E, L, J, W, for extended point transformation, first order problem; (2) E, L, W, for extended point transformation, second order problem; (3) L, W, Lie contact transformation, second order problem. E, under a_2, a_3 is independent of Kürschak in *Mathematische Annalen*, volume 55, where it is shown for problem of the nth order. E, L, W, under b_1, *cf.* Bolza, Calculus of variations, § 35.

14. In Mr. Graber's paper the cyclide is defined as a fourth order point surface consistent with the equations (1) $b^2 + c^2 + d^2 - ae = 0$ and (2) $\phi(a, b, c, d, e) = 0$, where (1) is the second degree equation which remains unchanged by the point transformations of ordinary sphere geometry, and (2) is any equation of the second degree. A classification of cyclidal surfaces on the basis of nodal conics, connectivity with the real surface, and surface deficiency formulæ, is then briefly outlined. The ordinary surfaces of the second degree are special cases of cyclides, and generalizations of value in classification may be obtained by extending the known properties of quadric surfaces to cyclides.

15. The second paper by Professor Dickson is devoted to a generalization of the important results obtained by Maillet (*Annales de Toulouse*, 1904) on the Galois group of the equation defining certain types of geometric configurations and the application to the number of real elements in the configuration. Employing a different method of attack, the paper establishes the generalizations quite simply, in contrast to the elaborate

analysis of Maillet. For the case of the 28 bitangents to a quartic curve, it is shown by group theory that the number of real bitangents must be 4, 8, 16 or 28, in accord with Schlafli's geometric analysis. Maillet's method does not exclude the impossible case of no real bitangents. For the case of the 27 straight lines on a cubic surface, 3, 7, 15 or 27 must be real, a result in accord with that of the geometers. The paper has been offered to the *Annals of Mathematics*.

16. Relations of order and continuity play relatively insignificant rôles among the classical theorems of synthetic projective geometry. These theorems can nearly always be stated in terms of incidence relations. Dr. Veblen has shown that practically the whole field covered by elementary courses in projective geometry can be developed from axioms of incidence together with the assumption that, if two projective ranges have their point of intersection self-corresponding, they are perspective. On the basis of these axioms, an analytic geometry can be developed in which the coordinates are marks of a field. Conversely, an analytic geometry whose coordinates are marks of an arbitrary field satisfies the axioms indicated. A slight modification of the theory is needed for a field with modulus 2. The case of finite fields is discussed in the paper of Dr. Bussey. A synthetic treatment of projective geometry according to this scheme is contained in a set of lecture notes on a course at the University of Chicago, compiled and mimeographed by Messrs. N. J. Lennes and R. L. Börger.

17. On account of the one to one correspondence between the totality of real numbers and the points of a straight line, there is a close connection between algebra and geometry, and consequently an advance in one often carries with it an advance in the other.

If the analogy is to hold when the algebraic domain is finite, there should be, in the realm of abstract geometry, a finite geometry to correspond to the algebra of every finite field. The paper by Dr. Veblen and Dr. Bussey exhibits a finite n-dimensional geometry ($n = 0, 1, 2, 3, \ldots$), corresponding to every finite field. The usual names point, straight line, plane, 3-space, . . ., and n-space are given to the geometries of 0, 1, 2, 3, . . ., and n dimensions respectively. The finite geometry corresponding to a field of order s is obtained by tak-

ing a linear homogeneous equation in $k+1$ variables as the equation of a $(k-1)$-space in k-dimensional geometry, the domain for the coefficients and variables being the field of order s. These geometries are also studied from the point of view of pure synthetic geometry. A study is made of curves and surfaces of the second order; also of the groups of the geometries. Finally certain derived tactical configurations are obtained among which are triple systems in t elements where $t = 2^k - 1$ or 3^k.

18. Frobenius and C. S. Peirce have shown that, in the domain of real numbers, the only linear associative algebras every element of which, except zero, possesses an inverse are quaternions and its subalgebras. Mr. Wedderburn shows that the Galois field is the only algebra of this type which possesses a finite number of elements.

19. Mr. Owens's paper is principally concerned with these two subjects: (*a*) The introduction of the plane ideal elements of projective geometry, independently of the parallel axiom, and of continuity. (*b*) A proof, independent of the parallel axiom and of continuity, that a certain form of the Desargues theorem is a sufficient condition that a plane satisfying order axioms may be a part of a projective three-space.

20. In the paper of Dr. Veblen and Mr. Wedderburn, it is pointed out that Hilbert's proof that Pascal's theorem is not deducible from Axioms I, II, and III of his "Grundlagen der Geometrie," is incorrect, namely the algebra of his non-pascalian geometry does not satisfy axiom 16, § 13. A proof of Hilbert's theorem is given by a modification of his geometry and a second proof that Pascal's theorem cannot be deduced from axioms I and III is obtained by means of an analytic geometry based on quaternions. Lastly, a new set of finite configurations, which may be regarded as non-desarguesian finite geometries, is discussed.

21. Dr. Veblen's note calls attention to certain connections between order relations and the existence of square roots in a number system. Among other ways of stating the relation is the following. Define -1 as a mark such that $-1+1=0$, a square as a mark for which there exists in the field considered a mark x such that $x.x = a$, and a not-square as a mark

which is not a square; one may add to any set of postulates defining a field these two: (α) -1 is a not-square. (β) The sum of two not-squares is a not-square. On defining $a < b$ to mean that $a - b$ is a not-square, the usual propositions about the symbol $<$ follow at once. If a continuity axiom is added, the system of postulates so obtained defines the real number system. This note is to be published in the *Transactions*.

22. The third paper by Professor Dickson deals with the generalization of the concept of hypercomplex number systems and of the precise definition of this generalized concept by means of independent postulates. The elements are $a = (a_1, \cdots, a_n)$, where the a_i are marks of a given field F. A system of such elements together with n^3 fixed marks γ_{ijk} form a number system if the following six postulates hold: (1) if a and b are elements of the system then $(a_1 + b_1, \cdots, a_n + b_n)$ is also an element; (2) the element $0 = (0, \cdots, 0)$ occurs in the system; (3) if 0 occurs, then to any element a corresponds an element a' of the system such that $a_i + a_i' = 0$ $(i = 1, \cdots, n)$; (4) if a and b are any two elements, then $(p_1, \cdots, p_n)$ is an element of the system, where $p_i = \Sigma a_j b_k \gamma_{jki}$; (5) the usual relations between the γ's assuring associativity of multiplication; (6) if $\tau_1, \cdots, \tau_n$ are marks of F such that $\tau_1 a_1 + \cdots + \tau_n a_n = 0$ for every a, then $\tau_1 = 0, \cdots, \tau_n = 0$. It is shown that n units linearly independent with respect to F can be determined. The paper is to appear in the July number of the *Transactions*.

J. W. YOUNG,
Secretary pro tem. of the Section.

EVANSTON, ILL.,
May 20, 1905.

A GENERAL THEOREM ON ALGEBRAIC NUMBERS.

BY PROFESSOR L. E. DICKSON.*

(Read before the American Mathematical Society, December 29, 1904.)

1. LET $r_1, \cdots, r_n$ belong to a field F and let

$$\rho^n = \sum_{i=1}^{n} r_i \rho^{n-i} \tag{1}$$

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be irreducible in F. A multiplication table

$$\rho^{n-i}\rho^{n-k} = \sum_{s=1}^{n} \gamma_{iks}\rho^{n-s} \qquad (i, k = 1, \cdots, n) \tag{2}$$

enables us to express the product of

$$x = \sum_{i=1}^{n} x_i\rho^{n-i}, \quad y = \sum_{k=1}^{n} y_k\rho^{n-k} \qquad (x_i, y_k \text{ in } F) \tag{3}$$

in the form $xy = z$,

$$z = \sum_{s=1}^{n} z_s\rho^{n-s}, \quad z_s \equiv \sum_{i,k}^{1,\cdots,n} \gamma_{iks}x_iy_k \quad (s = 1, \cdots, n). \tag{4}$$

THEOREM. *Each of the* $3n$ *determinants**

$$C_i \equiv |\gamma_{iks}|, \quad C'_k \equiv |\gamma_{iks}|, \quad C''_s = |\gamma_{iks}| \tag{5}$$

is not zero. We have the evaluations

$$\begin{gathered} C_i = (-1)^{i(n-i)}r_n^{n-i}, \quad C'_k = (-1)^{k(n-k)}r_n^{n-k}, \\ C''_1 = \theta_n, \quad C''_n = \theta_{n-1}r_n^{n-1}, \end{gathered} \tag{6}$$

where θ_m *is* $+1$ *if* m *is of the form* $4l$ *or* $4l+1$, *but* θ_m *is* -1 *if* m *is of the form* $4l+2$ *or* $4l+3$; *while for* $1 < s < n$, C''_s *is, aside from sign, the resultant of the equations*

$$\rho^{s-1} = \sum_{i=1}^{s-1} r_i\rho^{s-1-i}, \quad 0 = \sum_{i=s}^{n} r_i\rho^{n-i}, \tag{7}$$

obtained by splitting equation (1) *into two parts.*

2. That $C_i \neq 0$ and $C'_k \neq 0$ follows readily from the theory of higher complex numbers. For any y and z in $F(\rho)$, $y \neq 0$, there is an unique solution x of $xy = z$; for any x and z in $F(\rho)$, $x \neq 0$, there is an unique solution y of $xy = z$. Hence of the determinants

$$\Delta \equiv \left| \sum_{i=1}^{n} \gamma_{iks}x_i \right|_{k,s=1,\ldots,n}, \quad \Delta' \equiv \left| \sum_{k=1}^{n} \gamma_{iks}y_k \right|_{i,s=1,\ldots,n},$$

Δ vanishes only if $x_1 = \cdots = x_n = 0$, Δ' vanishes only if $y_1 = \cdots = y_n = 0$. But C_i is the coefficient of x_i^n in Δ, C'_k is the coefficient of y_k^n in Δ'.

3. From (1) we obtain

$$\rho^{n+j} = \sum_{s=1}^{n} b_{sj}\rho^{n-s}, \quad b_{sj} \equiv \sum_{i=0}^{j} r_{s+j-i}f_i, \tag{8}$$

* Sections of the cubic array $|\gamma_{iks}|$ by planes parallel to the faces.

where r_t $(t > n)$ is to be replaced by zero, and where

$$f_0 = 1, \quad f_1 = r_1, \quad f_2 = r_2 + r_1^2, \quad f_3 = r_3 + 2r_2r_1 + r_1^3,$$
$$f_4 = r_4 + 2r_3r_1 + r_2^2 + 3r_2r_1^2 + r_1^4,$$
$$f_5 = r_5 + 2r_4r_1 + 2r_3r_2 + 3r_3r_1^2 + 3r_2^2r_1 + 4r_2r_1^3 + r_1^5,$$

f_i being the sum of all terms of weight i in the r's with the number of arrangements of the r's as coefficients. Then

$$(9) \qquad f_{j+1} = \sum_{i=0}^{j} r_{1+j-i} f_i, \; f_{j+1} = b_{1j}.$$

A comparison of $(8)_1$ with (2) gives

$$(10) \quad \gamma_{iks} = b_{s\,n-i-k} \quad (k \leqq n-i), \quad \gamma_{iks} = \delta_{k\,n-i+s} \quad (k > n-i),$$

where

$$\delta_{kk} = 1, \; \delta_{kt} = 0 \text{ if } t \neq k.$$

4. Applying $(10)_1$ for the first $n-i$ rows and $(10)_2$ for the last i rows, we obtain for $C_i = |\gamma_{iks}|$ the expression

$$\begin{vmatrix} b_{1\,n-i-1} & b_{2\,n-i-1} & \cdots & b_{n\,n-i-1} \\ \cdots & \cdots & \cdots & \cdots \\ b_{11} & b_{21} & \cdots & b_{n1} \\ b_{10} & b_{20} & \cdots & b_{n0} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \sigma_i \begin{vmatrix} b_{i+1\,n-i-1} & \cdots & b_{n\,n-i-1} \\ \cdots & \cdots & \cdots \\ b_{i+1\,1} & \cdots & b_{n\,1} \\ b_{i+1\,0} & \cdots & b_{n\,0} \end{vmatrix},$$

where $\sigma_i = (-1)^{i(n-i)}$. Now by $(8)_2$, $b_{s0} = r_s$, $b_{s1} - f_1 r_s = r_{s+1}$, $b_{s2} - f_1 r_{s-1} - f_2 r_s = r_{s+2}, \cdots$ Thus,

$$C_i = \sigma_i \begin{vmatrix} r_n & 0 & 0 & \cdots & 0 & 0 \\ r_{n-1} & r_n & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{i+2} & r_{i+3} & r_{i+4} & \cdots & r_n & 0 \\ r_{i+1} & r_{i+2} & r_{i+3} & \cdots & r_{n-1} & r_n \end{vmatrix} = \sigma_i r_n^{n-i}.$$

Since $\gamma_{iks} = \gamma_{kis}$, we have $C_k' = |\gamma_{kis}|_{s=1,\ldots,n} = C_k = \sigma_k r_n^{n-k}$.

5. Similarly, C_s'' equals

$$\begin{vmatrix}
b_{sn-2} & b_{sn-3} & b_{sn-4} & \cdot & \cdot & \cdot & \cdot & b_{s1} & b_{s0} & 0 \\
b_{sn-3} & b_{sn-4} & \cdot & \cdot & \cdot & \cdot & b_{s1} & b_{s0} & 0 & 0 \\
b_{sn-4} & \cdot & \cdot & \cdot & \cdot & b_{s1} & b_{s0} & 0 & 0 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & b_{s1} & b_{s0} & 0 & 0 & \cdot & 0 & 1 \\
\cdot & \cdot & b_{s1} & b_{s0} & 0 & 0 & \cdot & 0 & 1 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
b_{s1} & b_{s0} & 0 & 0 & \cdot & 0 & 1 & 0 & \cdot & \cdot \\
b_{s0} & 0 & 0 & \cdot & 0 & 1 & 0 & \cdot & \cdot & 0 \\
0 & 0 & \cdot & 0 & 1 & 0 & \cdot & \cdot & 0 & 0
\end{vmatrix}$$

All the elements in a parallel to the left-hand diagonal are equal. The parallel with the elements 1 joins the sth element of the last row with the sth element of the last column.

In C_1'' the elements of the left-hand diagonal are all 1, and every element below this diagonal is zero. Hence $C_1'' = \theta_n = \pm 1$. In C_n'' every element in and below the left-hand diagonal is zero except the element 1 at the intersection of the last row and last column. Hence $C_n'' = \theta_{n-1} b_{s0}^{n-1}$.

For $1 < s < n, C_s'' = (-1)^{n-s} D_{sn}$, where D_{sn} is the minor of the element 1 in the last column. We proceed to the proof that D_{sn} is the eliminant by Sylvester's dialytic method of equations (7). The nature of the general proof will be clear from the following two illustrations:

For $n = 5$, $s = 2$, D_{sn} is

$$\begin{vmatrix}
b_{23} & b_{22} & b_{21} & b_{20} \\
b_{21} & b_{20} & 0 & 1 \\
b_{20} & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{vmatrix}, \qquad \begin{aligned} b_{20} &= r_2, \\ b_{21} &= r_3 + r_2 f_1, f_1 = r_1, \\ b_{22} &= r_4 + r_3 f_1 + r_2 f_2, \\ b_{23} &= r_5 + r_4 f_1 + r_3 f_2 + r_2 f_3. \end{aligned}$$

Multiply the fourth column by $-f_1, -f_2, -f_3$ and add to the third, second, first columns, respectively. In the resulting determinant, multiply the third column by $-f_1$ and $-f_2$, and add to the second and first columns, respectively. The first and second elements of the second row are now (§ 3)

$$r_3 + r_2 r_1 - f_3 + f_2 r_1 = 0,\ r_2 - f_2 + r_1^2 = 0.$$

Multiply the new second column by $-f_1$ and add to the first. There results

$$D_{25} = \begin{vmatrix} r_5 & r_4 & r_3 & r_2 \\ 0 & 0 & -r_1 & 1 \\ 0 & -r_1 & 1 & 0 \\ -r_1 & 1 & 0 & 0 \end{vmatrix},$$

the eliminant of $r_5 + r_4\rho + r_3\rho^2 + r_2\rho^3 = 0,\ -r_1 + \rho = 0.$

For $n = 4$, $s = 3$, D_{34} is

$$\begin{vmatrix} b_{32} & b_{31} & b_{30} \\ b_{31} & b_{30} & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad \begin{aligned} b_{30} &= r_3, \\ b_{31} &= r_4 + r_3 f_1, \\ b_{32} &= r_4 f_1 + r_3 f_2, \end{aligned}$$

the term r_5 in b_{32} being dropped since $5 > n$ (§ 3). Multiply the third column by $-f_1$ and $-f_2$ and add to the second and first columns, respectively. Multiply the new second column by $-f_1$ and add to the first. There results

$$D_{34} = \begin{vmatrix} 0 & r_4 & r_3 \\ r_4 & r_3 & 0 \\ -r_2 & -r_1 & 1 \end{vmatrix},$$

the eliminant of $r_4 + r_3\rho = 0,\ -r_2 - r_1\rho + \rho^2 = 0.$

CHICAGO, *December* 8, 1904.

ON THE DEFORMATION OF SURFACES OF TRANSLATION.

BY DR. L. P. EISENHART.

(Read before the American Mathematical Society, February 25, 1905.)

IN the January number of the BULLETIN* Dr. Burke Smith states the following theorem: The only non-developable surfaces of translation which may be deformed so that their gener-

* "On the deformation of surfaces of translation," p. 187.

ating lines remain generating lines throughout the deformation are the minimal surfaces and those surfaces of translation whose two systems of generating curves lie in perpendicular planes. This theorem is not true unless it be understood that the deformation is continuous, and not merely that the foregoing kinds of surfaces of translation are the only ones applicable to surfaces of translation with the generating lines in correspondence. In fact, Bianchi * uses the term "deformable in a continuous manner" in the theorem which is the basis of the above-mentioned theorem.

It has been known for a long time that if a minimal surface referred to its lines of length zero be defined by equations of the form

$$(1) \qquad x = U_1 + V_1, \quad y = U_2 + V_2, \quad z = U_3 + V_3,$$

where U_1, U_2, U_3; V_1, V_2, V_3 are functions of u and v respectively, the surfaces defined by

$$(2) \qquad x_1 = e^{ia}U_1 + e^{-ia}V_1, \quad y_1 = e^{ia}U_2 + e^{-ia}V_2,$$
$$z_1 = e^{ia}U_3 + e^{-ia}V_3$$

are minimal surfaces applicable to the given ones for all values of the constant a. In 1878, Bianchi † considered the surfaces of translation generated by an invariable plane curve subjected to a translation in which a point of the generator describes a curve lying in a plane perpendicular to the plane of the former; he showed that any surface of this kind can be defined by

$$(3) \quad x = \int \sqrt{1 - U'^2}\, du, \quad y = \int \sqrt{1 - V'^2}\, dv, \quad z = U + V,$$

when U and V are arbitrary functions of u and v respectively and the primes denote differentiation; furthermore, he remarked that the surfaces given by

$$(4) \qquad x_1 = \int \sqrt{1 - U'^2 e^{2a}}\, du, \qquad y_1 = \int \sqrt{1 - V'^2 e^{-2a}}\, dv,$$
$$z_1 = e^a U + e^{-a} V,$$

when α denotes a constant, are surfaces of the same kind and

* Lezioni, II, p. 83; German translation, p. 337.

† "Sopra la deformazione di una classe di superficie," *Giornale di Matematiche*, vol. 16 (1878), p. 267.

are applicable to the given surface for all values of α.* It is now our purpose to show how one can obtain from the foregoing surfaces large numbers of pairs of surfaces of translation applicable to one another with correspondence of the generators, but not in a continuous manner.

To this end we make use of a theorem of Adam: † Let $S(x, y, z)$ and $S_1(x_1, y_1, z_1)$ be any two applicable surfaces, it is readily verified that the two surfaces S' and S_1', defined by

$$(5)\begin{cases} x' = x + h(z + z_1) - k(y + y_1), & x_1' = x_1 - h(z + z_1) + k(y + y_1), \\ y' = y + k(x + x_1) - g(z + z_1), & y_1' = y_1 - k(x + x_1) + g(z + z_1), \\ z' = z + g(y + y_1) - h(x + x_1), & z_1' = z_1 - g(y + y_1) + h(x + x_1), \end{cases}$$

where g, h, k denote three arbitrary constants, are applicable for all values of these constants.

Before proceeding to the use of this theorem in our particular study, we wish to call attention to a general property of these pairs of applicable surfaces. From (5) it follows that

$$x' + x_1' = x + x_1, \quad y' + y_1' = y + y_1, \quad z' + z_1' = z + z_1.$$

Hence through the mean point of the join of corresponding points on S and S_1 there passes the line joining the points, corresponding to the former, on each pair S' and S_1' defined by (5); and, moreover, the point of intersection is the mean point of these segments also.

In order to put the equations of a surface of translation with generators in perpendicular planes in the form (3), it is necessary usually to effect two quadratures. But these can be done away with; for if we write the equations in the form

$$(6) \qquad x = U, \quad y = V, \quad z = U_1 + V_1,$$

where U, U_1; V, V_1 are arbitrary functions of u and v respectively, the surface defined by

* Pirondini, "Sulle superficie di translatione," *Annali di Matematica*; ser. 3, vol. 17 (1889), p. 225, proposes to find all surfaces of translation which are applicable to surfaces of the same kind with correspondence of the generators; in formulating the equations of condition he failed to take account of the exceptional case which leads to the minimal surfaces; he finds the surfaces (3) and two other groups of surfaces applicable but not in a continuous manner.

† "Sur la déformation des surfaces", *Bull. S. M. F.*, vol. 23 (1895), p. 106.

$$x_1 = \int \sqrt{U'^2 - (e^{2a} - 1)U_1'^2}\, du,$$

(7)

$$y_1 = \int \sqrt{V'^2 - (e^{-2a} - 1)V_1'^2}\, dv, z_1 = U_1 e^a + V_1 e^{-a}$$

is applicable to the given surface for all values of the constant a.* For the sake of brevity, we put

$$U_2 = \int \sqrt{U'^2 - (e^{2a} - 1)U_1'^2}\, du,$$

(8)

$$V_2 = \int \sqrt{V'^2 - (e^{-2a} - 1)V_1'^2}\, dv,$$

and then substitute the above values in (5). This gives

$$(9)\quad \begin{cases} x' = [U + hU_1(1 + e^a)] + [h(1 + e^{-a})V_1 - k(V + V_2)], \\ y' = [k(U + U_2) - g(1 + e^a)U_1] + [V - g(1 + e^{-a})V_1], \\ z' = [U_1 - h(U + U_2)] + [V_1 + g(V + V_2)], \end{cases}$$

and

$$(10)\quad \begin{cases} x_1' = [U_2 - h(1 + e^a)U_1] + [-h(1 + e^{-a})V_1 + k(V + V_2)], \\ y_1' = [-k(U + U_2) + g(1 + e^a)U_1] + [V_2 + g(1 + e^{-a})V_1], \\ z_1' = [U_1 e^a + h(U + U_2)] + [V_1 e^{-a} - g(V + V_2)]. \end{cases}$$

Evidently these equations define surfaces of translation, and from the theorem of Adam it follows that they are applicable for all values of the constants a, g, h, k; but as these constants appear in both sets of equations, the surfaces are applicable in a discontinuous manner.

An example of a surface of translation of the type (6) is afforded by the paraboloids, whose equations may be written thus

$$x = u, \quad y = v, \quad z = \frac{u^2}{2a} + \frac{v^2}{2b}.$$

*Mlodzieowski, "Sur la déformation des surfaces", *Bull. des Sciences Math.*, ser. 2, vol. 15 (1891), p. 97.

Now

$$U_2 = \int \sqrt{1 - (e^{2a} - 1)\frac{u^2}{a_2}}\, du, \quad V_2 = \int \sqrt{1 - (e^{-2a} - 1)\frac{v^2}{b^2}}\, dv;$$

these quadratures can be effected readily and depend for their results upon the values of a. Hence we can find not only an infinity of surfaces of translation applicable to the paraboloids, but also a four parameter assemblage of pairs of surfaces of translation applicable to one another without continuity in the deformation; and this can be done without further quadrature.

In order that a surface defined by equations of the form (9) be such that the curves $u = \text{const.}$ be plane, it is necessary and sufficient that there be three constants a, b, c such that $ax' + by' + cz'$ be independent of v. If this is to be true for arbitrary forms of the functions V and V_1 it is necessary and sufficient that the following relations obtain:

$$-ak + b + cg = 0, \quad a(1 + e^{-a})h - bg(1 + e^{-a}) + c = 0,$$
$$ak - cg = 0,$$

which reduce to

$$ak = cg, \quad b = 0, \quad ah(1 + e^{-a}) + c = 0,$$

so that either

$$b = h = c = k = 0, \tag{11}$$

or

$$b = 0, \; h \neq 0, \; k = -hg(1 + e^{-a}). \tag{11'}$$

In the former case the planes of the curves $u = \text{const.}$ are parallel to the yz plane, in the latter to the plane whose equation is

$$X - h(1 + e^{-a})Z = 0. \tag{12}$$

In a similar manner the conditions that the curves $v = \text{const.}$ lie in parallel planes for all values of U and U_1 reduce to

$$a' = g = c' = k = 0, \tag{13}$$

or

$$a' = 0, \; g \neq 0, \; k = hg(1 + e^{a}), \tag{13'}$$

so that in the first case the planes are parallel to the xz plane and in the second to the plane

$$Y + g(1 + e^a) Z = 0. \tag{14}$$

From the preceding investigation it follows that in order that both families of curves be plane, it is necessary that

$$k = -hg(1 + e^{-a}) = hg(1 + e^a),$$

from which it follows that h or g is equal to zero, or $e^a = e^{-a} = 1$. For the latter case the surfaces S and S_1 are symmetric with respect to the xy plane, as are S' and S_1' also, the coördinates of S' being

$$x' = U, \quad y' = V, \quad z' = U_1 - 2hU + V_1 + 2gV;$$

evidently this surface is of the same kind as S. When $g = 0$ (the case $h = 0$ being similar to it), the curves $v = \text{const.}$ lie in planes parallel to the xz plane and the curves $u = \text{const.}$ in planes parallel to the plane (12). Hence the planes of the curves are perpendicular, so that we have the theorem:

*When the parametric curves in both systems are plane for the surfaces S', the planes are perpendicular to one another.**

From (10) we find that in order that the curves $v = \text{const.}$ on S_1' be plane, it is necessary and sufficient that either

$$g = k = 0 \tag{15}$$

or

$$hg(1 + e^{-a}) + k = 0. \tag{15'}$$

In the former case the planes of the curves are parallel to the xz plane and in the latter to

$$Y - g(1 + e^{-a}) Z = 0.$$

Similar results follow for the curves $u = \text{const.}$ on S_1', hence the theorem:

When one family of parametric curves on S' are plane, the curves on S_1' corresponding to the other family on S' are parallel.

From (12) and the above equation it follows that these planes are perpendicular only in case h or g is equal to zero.

It is known that any real minimal surface can be defined by equations (1), where U_1, U_2, U_3 have the values

*Cf. Adam, "Théorème sur la déformation des surfaces de translation," *Bull. S. M. F.* vol. 23 (1895), p. 204.

$$
\begin{aligned}
U_1 &= \frac{1-u^2}{2} f''(u) + uf'(u) - f(u), \\
U_2 &= i\frac{1+u^2}{2} f''(u) - iuf'(u) + if(u), \\
U_3 &= uf''(u) - f'(u),
\end{aligned} \tag{16}
$$

and V_1, V_2, V_3 are those functions of the conjugate variable v which are obtained when u is replaced by v and $f(u)$ by the conjugate function of v.

When the expressions (1) and (2) are substituted in (5), we get for S'

$$
\left\{
\begin{aligned}
x' &= [U_1 + h(1+e^{ia})U_2 - k(1+e^{ia})U_2] \\
&\qquad + [V_1 + h(1+e^{-ia})V_3 - k(1+e^{-ia})V_2], \\
y' &= [U_2 + k(1+e^{ia})U_1 - g(1+e^{ia})U_3] \\
&\qquad + [V_2 + k(1+e^{-ia})V_1 - g(1+e^{-ia})V_3], \\
z' &= [U_3 + g(1+e^{ia})U_2 - h(1+e^{ia})U_1] \\
&\qquad + [V_3 + g(1+e^{-ia})V_2 - h(1+e^{-ia})V_1].
\end{aligned}
\right. \tag{17}
$$

Here also is a four-fold assemblage of surfaces of translation; and we can find directly by means of (5) a surface of translation applicable to each one of them. Since $(1+e^{ia})U_3$ and $(1+e^{-ia})V_3$ are conjugate imaginaries, the constant h must be real if S' is to be real; similarly for g and k.

After the manner pursued in the study of equations (10) we find that the necessary and sufficient condition that the curves $v = \text{const.}$ on the surface S' be plane for any form of the function f in (16) is that there exist three constants a, b, c satisfying the equations

$$
\begin{aligned}
a + k(1+e^{ia})b - h(1+e^{ia})c &= 0. \\
k(1+e^{ia})a - b - g(1+e^{ia}) &= 0, \\
h(1+e^{ia})a - g(1+e^{ia})b + c &= 0.
\end{aligned}
$$

The determinant of these equations is equal to

$$1 + (1+e^{ia})^2(g^2+h^2+k^2).$$

Since g, h, k are real, this determinant cannot vanish for any real value of α. Hence there do not exist for each form

of f surfaces S' with the curves $v = \text{const.}$ plane. Moreover, if the values from (16) be substituted in (17) and these equations be multiplied by any real constants a, b, c, the solution of a differential equation of the second order gives a function $f(u)$ such that $ax' + by' + cz'$ is independent of u. But since V_1, V_2, V_3 are conjugate imaginaries of U_1, U_2, U_3, the expression is free of v also, and consequently the surface is plane. Excluding planes from the discussion, we have that *the parametric curves on the surfaces S' defined by* (17) *are curves of double curvature.*

After stating the theorem involving equations (5), Adam remarks that the theorem is true whatever be the relative position of S and S_1; this change of relative position can be effected in the most general way by keeping S fixed and subjecting S_1 to a general rotation in space, which amounts to replacing x_1, y_1, z_1, by

$$(18)\quad a_1x_1 + b_1y_1 + c_1z_1,\ a_2x_1 + b_2y_1 + c_2z_1,\ a_3x_1 + b_3y_1 + c_3z_1,$$

where $a_1, b_1, \cdots, c_3$ are the coefficients of an orthogonal substitution such that

$$(19)\quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = +1.$$

When this substitution is made in (5), there result

$$(20)\quad \begin{cases} x' = x + hz - ky + (ha_3 - ka_2)x_1 + (hb_3 - kb_2)y_1 + (hc_3 - kc_2)z_1, \\ y' = y + kx - gz + (ka_1 - ga_3)x_1 + (kb_1 - gb_3)y_1 + (kc_1 - gc_3)z_1, \\ z' = z + gy - hx + (ga_2 - ha_1)x_1 + (gb_2 - hb_1)y_1 + (gc_2 - hc_1)z_1, \end{cases}$$

and

$$(21)\quad \begin{cases} x_1' = ky - hz + (a_1 - ha_3 + ka_2)x_1 + (b_1 - hb_3 + kb_2)y_1 \\ \qquad\qquad + (c_1 - hc_3 + kc_2)z_1, \\ y_1' = gz - kx + (a_2 - ka_1 + ga_3)x_1 + (b_2 - kb_1 + gb_3)y_1 \\ \qquad\qquad + (c_2 - kc_1 + gc_3)z_1, \\ z_1' = hx - gy + (a_3 - ga_2 + ha_1)x_1 + (b_3 - gb_2 + hb_1)y_1 \\ \qquad\qquad + (c_3 - gc_2 + hc_1)z_1. \end{cases}$$

On substituting for x, y, z; x_1, y_1, z_1 the values (6) and (7),

we obtain surfaces of translation, applicable to one another and depending upon seven arbitrary parameters.

The conditions that there exist three constants a, b, c such that for all values of V and V_1 the curves $u = \text{const.}$ lie in the planes $aX + bY + cZ = d$ are

$$ak - b - cg = 0,$$

$$a(hb_3 - kb_2) + b(kb_1 - gb_3) + c(gb_2 - hb_1) = 0,$$

$$a[h + (hc_3 - kc_2)e^{-a}] + b[-g + e^{-a}(kc_1 - gc_3)] + c[1 + e^{-a}(gc_2 - hc_1)] = 0.$$

Equating to zero the determinant of these equations, we get in consequence of (19) the following equations of conditions

$$(22) \quad (k^2 + h^2)b_1 - (k + gh)b_2 + (h - kg)b_3 = h(ga_1 + ha_2 + ka_3)e^{-a},$$

and the planes of the curve are parallel to the plane

$$[(h + gk)b_1 - gb_2 - g^2b_3]x + h(kb_1 - gb_3)Y + [k^2b_1 - kb_2 + (h - kg)b_3]Z = 0.$$

As in the particular case previously considered, equation (22) is the condition also that the curves $v = \text{const.}$ on S_1' lie in parallel planes.

Another seven parameter aggregate of pairs of applicable surfaces of translation is found when the values from (1) and (2) are substituted in equations (20) and (21).

PRINCETON UNIVERSITY,
February, 1905.

THE GROUPS OF ORDER 2^m WHICH CONTAIN AN INVARIANT CYCLIC SUBGROUP OF ORDER 2^{m-2}.

BY PROFESSOR G. A. MILLER.

HALLET * has recently called attention to the fact that Burnside omits one group in his enumeration of the non-abelian groups of order 2^m which contain an invariant cyclic subgroup

* BULLETIN, vol. 11 (1905), p. 318; *Science*, vol. 21 (1905), p. 176.

A of order 2^{m-2} such that the corresponding quotient group is cyclic. There are six such groups which do not include an operator of order 2^{m-1} while Burnside gives only five of them.* There are however only eight possible non-abelian groups of order 2^m which do not include an operator of order 2^{m-1} and give rise to a non-cyclic quotient group with respect to A, while Burnside's statement that the total number of groups in question is fourteen implies that there are nine such groups.

It seems desirable to give the characteristic properties of all the groups in question in order that the reader of Burnside may be able to verify the double error with ease and also to complete the list of the groups of order 2^m which include operators of order 2^{m-2}. The ten groups given in § 1 are included in the more extensive enumeration of all the groups of order 2^m which include the abelian group of type $(m-2, 1)$.† In particular, the group given by Hallet is G_{16} of this list. It is, however, somewhat difficult in several instances, to determine which of the groups of this list contain an invariant cyclic subgroup of order 2^{m-2} since the groups were determined from a different standpoint. On this account it seems desirable to include these groups in the present enumeration.

For detailed explanation of the method employed to prove that no other groups exist we would refer to the list mentioned in the preceding paragraph and also to a brief article in the *American Journal*, volume 24 (1902), page 395, "On a method of constructing all the groups of order p^m." In what follows it will be assumed that the groups in question are non-abelian and of order 2^m, include A, and do not include any operator of order 2^{m-1}.

Each of the groups under consideration contains at least one subgroup of order 2^{m-1} including A. Such a subgroup will be denoted by H while G will be used to represent the entire group. The symbol $G-H$ will be employed to represent the operators of G which are not also in H. The groups will be considered under two headings, according as H is abelian or non-abelian. Those under each heading will be subdivided into two categories, according as G/A is non-cyclic or cyclic.

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* Burnside, Theory of groups of finite order, 1897, p. 80. It is assumed throughout this article that $m > 5$. For smaller values of m there are exceptions.

† *Transactions of the Amer. Math. Society*, vol. 2 (1901), p. 259.

§ 1. *Groups in which H is abelian.*

The type of H is $(m-2, 1)$ and the subgroup I_1 of its group of isomorphisms, which is composed of all the operators which transform A into itself, is the direct product of the group of isomorphisms of A and an operator of order two. Hence I_1 is abelian and of type $(m-4, 1, 1)$. The operators in $G-H$ must transform H according to an operator of order 2 contained in I_1. If t represents such an operator G/A is cyclic or non-cyclic as t^2 is in $H-A$ or in A. We begin with the latter case.

As H contains two cyclic groups of order 2^{m-2} which are invariant under G, it is necessary to consider only one from each of the two pairs of operators of order 2 in I_1 which are conjugate under the group of isomorphisms of H.* That is, we have to consider only five of the seven operators of order two contained in I_1, viz., the three which transform each operator of H into the same power, the one which is commutative with every operator of A, and the one which transforms each operator of A into its inverse while it transforms the operators of $H-A$ into their inverses multiplied by the operator of order 2 contained in A. Each of the last two has two conjugates under the group of isomorphisms of H.

The two groups (G_1, G_2) which transform each operator of H into its inverse contain respectively 2^{m-1} operators of order 2, or 2^{m-1} operators of order 4 in addition to H. The group G_3 which transforms each operator of H into its $2^{m-3}+1$ power is conformal with the abelian group of type $(m-2, 1, 1)$. Its commutator subgroup is the group of order 2 contained in A. The group G_4 which transforms each operator of H into its $2^{m-3}-1$ power is composed of H and 2^{m-2} operators of each of the orders 2 and 4. These four groups are clearly distinct since no two of them are conformal.

The group G_5 which involves an invariant operator of order 2^{m-2} is also conformal with the abelian group of type $(m-2, 1, 1)$. It cannot be simply isomorphic with G_3 since the latter does not involve any invariant operator of order 2^{m-2}. Moreover, the operators whose orders divide 2^{m-3} constitute an abelian subgroup of G_3, while this is not the case with the operators of G_5. The remaining group G_6 transforms each operator of A into its inverse while it transforms the operators of $H-A$ into

* *Transactions of the Amer. Math. Society*, vol. 2 (1901), p. 260.

their inverses multiplied by the operator of order two contained in A. It is conformal with G_4, but contains invariant operators of order 4, while G_4 does not have this property. It may also be distinguished from G_4 by the fact that it transforms operators of order 2^{m-2}into their inverses. Hence *there are just six groups in which H is abelian and G/A is non-cyclic.*

The necessary and sufficient condition that G/A is cyclic is that t^2 is in $H - A$. The group G_7 which transforms each operator of H into its inverse is conformal with G_2. The squares of the operators in $G_7 - H$ are not powers of operators of higher order as is the case with the operators of $G_2 - H$. The group G_8 which transforms each operator of H into its $2^{m-3} + 1$ power is conformal with the abelian group of type $(m - 2, 2)$. Its commutator subgroup is of order 2 and all the operators of H whose orders divide 2^{m-3} are invariant. Its group of cogredient isomorphisms is the four group. The group G_9 which transforms each operator of H into its $2^{m-3} - 1$ power is conformal with G_2 and G_7. It is not identical with G_2 since the square of the operators in $G_9 - H$ are not powers of operators of higher order. It is not identical with G_7, since it does not transform its operators of highest order into their inverses, and also because the squares of the operators in $G_9 - H$ give two distinct operators of order two.

Only one case remains, viz., when G transforms the operators of A into their inverses and the operators of $H - A$ into their inverses multiplied by the operator of order 2 contained in A. In this case there is a group G_{10} which contains only operators of order 8 in addition to H. This is the group given by Hallet. It may be defined as the group generated by two operators of order 8 having a common square and such that the product of one of these operators into the inverse of the other is of order 2^{m-2} and generates the fourth power of one of the two generators of order 8. As all the possible cases have now been considered *there are just four groups in which H is abelian and G/A is cyclic.* The groups $G_1, G_2, \cdots, G_{10}$ of the present list are simply isomorphic respectively to $G_5, G_6, G_1, G_3, G_{10}, G_{15}, G_7, G_2, G_4, G_{16}$ of the *Transactions* list mentioned above.

§ 2. *Groups in which H is non-abelian.*

We shall again begin with the case when G/A is non-cyclic. As such a G contains three non-abelian subgroups of order

2^{m-1} which include A, it must transform A according to the three different operators of order 2 in its group of isomorphisms. Hence we may assume without loss of generality that H is conformal with the abelian group of type $(m-2, 1)$. Furthermore, it may be assumed that t transforms each operator of A into its inverse. As t^2 is in A, t is either of order 2 or of order 4. In the former case the group G_{11} generated by H and t contains 3.2^{m-2} operators of order 2 and 2^{m-2} of order 4 in addition to H, and in the latter group G_{12} the number of additional operators of order 4 is 3.2^{m-2} while the number of those of order 2 is 2^{m-2}.

It is easy to see that these are the only groups for which G/A is non-cyclic. Such a G transforms H according to the subgroup of order 8, containing only operators of order 2 besides the identity, in the group of isomorphisms of H. The subgroup I_2 of this group of isomorphisms which transforms A into itself is the direct product of the group of isomorphisms of A and an operator of order 2. Hence it contains only seven operators of order 2 and the way in which t transforms H may be regarded as determined in the above manner. As neither of these groups contains an abelian subgroup of order 2^{m-1} they are distinct from those determined in the preceding section. It remains to consider the case when G/A is cyclic.

As the H of such a G transforms A according to the square of an operator in its group of isomorphisms, it is the same group as in the preceding case. Since t^2 is in $H-A$, t is commutative with some of the operators of $H-A$. Moreover, if t transforms H according to one of the operators of order 4 in I_2 the operators in $G-H$ transform H according to four distinct operators of order 4 in I_2, since the group of cogredient isomorphisms of H is of order 4. As I_2 contains only eight operators of order 4, it is necessary to consider only two cases. In one of these t is commutative with every operator of H whose order divides 2^{m-4}. The resulting group G_{13} is conformal with G_8, but cannot be identical with this group since it contains no abelian subgroup of order 2^{m-1}. The commutator subgroup of G_{13} is the cyclic group of order 4 contained in A.

In the remaining case it may be assumed that t transforms into its inverse each operator of H whose order divides 2^{m-4} while it transforms the operators of highest order in H into their inverses multiplied by an operator of order four contained in A. The resulting group G_{14} contains 2^{m-2} operators of each

of the orders 4 and 8. Hence, *there are just four groups in which H is necessarily non-abelian.* In two of these G/A is cyclic while the quotient group is non-cyclic in the other two.

The total number of the non-abelian groups of order 2^m which contain an invariant cyclic subgroup of order 2^{m-2}, but no such subgroup of order 2^{m-1} is therefore fourteen. The last four were explicitly excluded from my list of these groups which do not contain an abelian subgroup of order 2^{m-1} including A* since Burnside had considered this subject. Knowing that Burnside gave the correct number of these groups I failed to observe the compensating errors. It may be added that the title of Hallet's paper as given in both reports noted above is misleading, since every possible group of order 2^m contains an invariant subgroup of order 2^{m-2}.

STANFORD UNIVERSITY,
March, 1905.

GALILEO AND THE MODERN CONCEPT OF INFINITY.

BY DR. EDWARD KASNER.

(Read before the American Mathematical Society, February 27, 1904.)

THE definition of an infinite assemblage, as one in which a part exists which may be put into one-to-one correspondence with the whole, recognized as fundamental in recent discussions by mathematicians and philosophers, is usually associated with the names of Bolzano, Cantor, and Dedekind. The object of this note is to call attention to a passage from Galileo which is of significance in this connection.

The passage in question appears incidentally in the work which contains Galileo's most permanent contribution to science, the foundations of dynamics; namely, the Discorsi e dimonstrazioni matematiche of 1638,† often referred to as the

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* *Transactions of the Amer. Math. Society*, vol. 3 (1902), p. 385.

† The full title, taken from a copy of the first edition in the Columbia University library, is as follows:

Discorsi | e | Dimostrazioni | matematiche, | intorno à due nuoue scienze | Attenenti alla | Mecanica & i Movimenti Locali, | del Signor | Galileo Galilei Linceo, | Filosofo e Matematico primario del Serenissimo | Grand Duca di Toscana. | Con vna Appendice del centro di grauità d'alcuni Solidi. | In Leida, | Appresso gli Elsevirii M.D.C. XXXVIII.

Dialogues on motion. It is written, partly in Italian, partly in Latin, in a style comparable with that of Plato. The interlocutors are the same as those in the better known dialogues on the ptolemaic and copernican systems, published six years earlier. Salviati, in general, represents the author's own views; Sagredo plays the rôle of the intelligent and appreciative critic; and Simplicio is the exponent of aristotelian scholastic philosophy.

The "first day" of the Discorsi opens with a discussion of the divisibility and continuity of matter and space. This leads to a consideration of the number of points on a straight line. Simplicio remarks that under the assumption of infinite divisibility we should be forced to admit that one infinity may be larger than another, since a long segment contains more points than a short one, though both would be infinite. We now translate quite literally.*

Salv. These difficulties arise because we with our finite mind discuss the infinite, attributing to the latter properties derived from the finite and limited. This, however, is not justifiable; for the attributes great, small, and equal are not applicable to the infinite, since one cannot speak of greater, smaller, or equal infinities. An example occurs to me which I shall refer to your consideration, Signor Simplicio, since it was you who started the discussion.

I take it for granted that you know which numbers are squares and which are not.

Simpl. I am aware of the fact that a square number arises through the multiplication of any number by itself; for example, 4 and 9 are square numbers formed from 2 and 3.

Salv. Excellent. You remember also that just as the products are called squares, the factors, that is, the numbers which are multiplied by themselves, are called sides or roots. The remaining numbers, which are not formed from two equal factors, are called non-squares. If then I state that all numbers, squares and non-squares taken together, are more numerous than the squares taken alone, that is an obviously correct proposition, is it not?

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* Page 32 in the original edition. An excellent German translation of the entire Discorsi, by von Oettingen, has appeared in Ostwald's Klassiker, Nos. 11, 24, 25; the passage mentioned begins on page 30 of No. 11. The most recent Italian version with commentary will be found in vol. 8 of the Edizione Nazionale of Galileo's works.

Simpl. It cannot be denied.

Salv. If now I ask how many squares are there, one can answer with truth, just as many as there are roots; for every square has a root, every root has a square, no square has more than one root, no root more than one square.

Simpl. Entirely correct.

Salv. Again, if I ask how many roots are there, one cannot deny that they are just as numerous as the complete number series, for there is no number which is not the root of some square. Admitting this, it follows that there are just as many squares as there are roots, since they are as numerous as the roots and every number is a root. Yet we said at the outset that all numbers are more numerous than all squares, since the majority of the former are non-squares. Indeed, the more numbers we take, the smaller is the proportion of squares; for up to 100 there are 10 squares, that is, one tenth are squares; up to 10000, one hundredth; up to 1000000, only one thousandth. Still up to an infinitely large number, granting we can conceive it, we were compelled to admit that there are just as many squares as numbers.

Simpl. What is to be our conclusion?

Salv. I see no escape except to say: the totality of numbers is infinite, the totality of squares is infinite, the totality of roots is infinite; the multitude of squares is not less than the multitude of numbers, neither is the latter the greater; and, finally, the attributes equal, greater and less are not applicable to infinite, but solely to finite quantities.*

The ensuing discussion is exceedingly interesting. It deals with the paradox "unity is infinity," the number of parts of a segment as distinguished from the number of points, and the straight line considered as a member of a family of circles. But enough has been cited for the present purpose.

COLUMBIA UNIVERSITY.

* Io non veggo che ad altra decisioni si possa venire, che à dire infiniti essere tutti i numeri, infiniti i quadrati, infinite le loro radici; nè la moltitudine de' quadrati esser minore di quella di tutti i numeri, nè questa maggior di quella; & in voltima conclusione gli attributi di eguale, maggiore, e minore non hauer luogo ne gl'infiniti, ma solo nelle quantità terminate.

NOTES.

AT the meeting of the London Mathematical Society held on April 13 the following papers were read: By Mr. P. W. WOOD, "On irreducible Jacobians of degree six"; by Mr. A. E. WESTERN, "On Fermat's numbers and the converse of Fermat's theorem"; by Professor A. E. H. LOVE, "On the strains that accompany bending"; by Dr. W. H. YOUNG, "On limiting sets of points in linear continua."

AT the meeting of the National academy at Washington, D. C., April 18–20, Professor M. I. PUPIN, of Columbia University, was elected to membership. No mathematical papers were presented at the meeting.

AT a recent meeting of the Institute of France, it was voted to make the following appropriations from the Debrousse legacy: 5000 francs for computing and compiling the lunar tables, 5000 francs to support the *Journal des savants*, and 3000 francs for preparing a catalogue of the works of Leibniz.

BEGINNING with the current volume 76, the *Proceedings* of the Royal Society of London are issued in two series, Series A containing the mathematical and physical papers, and Series B the biological papers. At the same time the size of the page has been increased to royal octavo, and the journal is printed in larger type.

AMONG the forthcoming books announced by Ginn and Company we notice the Theory of functions of real variables by Professor JAMES PIERPONT. The work consists of two volumes, the first of which is now in press; it will probably appear about August 1. The first volume is devoted to the foundations of the differential and integral calculus, and treats of the more important topics with great rigor and generality. The theory of irrational numbers introduced by Cantor and Dedekind is developed at considerable length, and the more elementary properties of point aggregates are freely used.

MR. M. M. DAWSON, consulting actuary of New York City, recently gave a series of lectures at the University of of Michigan in connection with the courses in insurance mathematics on the following subjects: pension funds, service pensions, mutual employees' insurance, fraternal insurance, and the history of the mathematics of insurance.

THE various American universities offer advanced courses in mathematics for the year 1905–1906 as follow:

BRYN MAWR COLLEGE. — By Professor CHARLOTTE A. SCOTT: Algebraic invariants, with applications, two hours; Modern analytic geometry, two hours. — By Mr. J. E. WRIGHT: Linear ordinary differential equations, two hours; Higher analysis, two hours. — By Dr. ISABEL MADDISON: Analytic geometry of space, one hour. The journal club will meet fortnightly.

UNIVERSITY OF CALIFORNIA. — By Professor I. STRINGHAM: Quaternions, three hours; Logic of mathematics, three hours; Seminar, two hours. — By Professor G. C. EDWARDS: Differential equations, three hours. — By Professor M. W. HASKELL: Analytic geometry, three hours; Algebraic forms and geometric transformations, three hours. — By Professor E. J. WILCZYNSKI: Projective differential geometry, three hours. — By Professor C. A. NOBLE: Calculus of variations (first half year) three hours; Theory of differential equations (second half year), three hours. — By Professor A. W. WHITNEY: Analytic geometry of three dimensions (second half year), three hours; Calculus of finite differences, two hours; Theory of probabilities, three hours. — By Professor D. N. LEHMER: Theory of equations, three hours. — By Dr. T. M. PUTNAM: Synthetic geometry, three hours; Theory of numbers, three hours. — By Dr. J. H. MCDONALD: Theory of functions of a real variable, three hours. — By Dr. B. L. NEWKIRK: Theory of complex functions, three hours.

UNIVERSITY OF CHICAGO, Summer quarter (June 17 to September 1, 1905). — By Professor O. BOLZA: Theory of functions of a complex variable; Selected chapters in the theory of functions. — By Professor H. MASCHKE: Advanced calculus; Differential geometry. — By Professor J. W. A. YOUNG: Critical review of secondary mathematics, for teachers. — By Professor L. E. DICKSON: Theory of numbers. Each course is five hours per week.

COLUMBIA UNIVERSITY. — By Professor T. S. FISKE: Advanced calculus, introduction to the theory of functions of a real variable, three hours; Theory of functions of a complex variable, three hours. — By Professor F. N. COLE: Theory of groups, three hours; Theory of plane curves, three hours. —

By Professor D. E. SMITH: History of mathematics, two hours. — By Professor JAMES MACLAY: Applications of the calculus to the theory of surfaces and curves in space, three hours. — By Professor C. J. KEYSER: Modern theories in geometry, three hours. — By Professor H. B. MITCHELL: Introduction to the calculus of quaternions, three hours. — By Dr. G. H. LING: Infinite series and products.—By Dr. EDWARD KASNER: Differential equations and continuous groups.

The department of physics offers a two years' cycle of courses in mechanics and mathematical physics as follows: By Professor M. I. PUPIN: Theory of the potential function, two hours; Partial differential equations of physics, two hours; Special problems, hydrokinetics, two hours. — By Professor A. P. WILLS; Mechanics, theory of elasticity, two hours; Electricity and magnetism, electromagnetic theory of light, two hours; thermodynamics, two hours, first half-year.

CORNELL UNIVERSITY. — By Professor L. A. WAIT: Differential calculus, two hours; Analytic geometry three hours. — By Professor G. W. JONES: Algebra, three hours. — By Professor J. MCMAHON: Theory of potential and spherical harmonics, three hours; Mechanics, two hours. — By Professor J. I. HUTCHINSON: Projective geometry, three hours; Infinite series and products, two hours. — By Professor V. SNYDER: Differential equations, two hours; Algebraic twisted curves, two hours. — By Dr. W. B. FITE: Definite integrals, two hours; Theory of functions, three hours. — By Dr. C. N. HASKINS: Theory of invariants, three hours; Calculus of variations, two hours; Differential equations, II, two hours. The Oliver mathematical club will meet fortnightly.

UNIVERSITY OF ILLINOIS. — By Professor S. W. SHATTUCK: Differential equations and calculus of variations, three hours.— By Professor A. N. TALBOT: Analytic mechanics, four hours. By Professor E. J. TOWNSEND: Theory of functions of a real variable, three hours; Solid analytic geometry (second semester), three hours; Seminar, three hours. — By Professor A. G. HALL: Potential function and spherical harmonics, three hours; Determinants, two hours. — By Professor H. L. RIETZ: Theory of invariants and higher plane curves, three hours. — By Professor J. STEBBINS: Least squares (first semester), two hours. — By Dr. H. L. COAR: Modern geometry and algebraic surfaces, three hours. — By Mr. E. L. MILNE: Mathematical theory of statistics (second semester), four hours.

INDIANA UNIVERSITY. — By Professor R. J. ALEY: Algebraic invariants (fall and winter terms), three hours; Theory of numbers (spring term), three hours; Ordinary differential equations (fall term), five hours. — By Professor S. C. DAVISSON: Modern geometry (fall and winter), two hours; Theory of surfaces (winter and spring), three hours. — By Professor D. A. ROTHROCK: Partial differential equations (fall aud winter), three hours; Theory of functions (winter and spring), three hours. — By Professor U. S. HANNA: Groups of substitutions and Galois's theory (fall and winter), two hours.

STATE UNIVERSITY OF IOWA. — By Professor L. G. WELD: The general theory of functions, two hours; Least squares (first semester), two hours; Elliptic integrals and functions (second semester), two hours; Fourier's series and spherical harmonics, two hours. — By Professor J. V. WESTFALL: Advanced calculus, three hours; Differential equations from the standpoint of the theory of functions, two hours. — By Dr. E. L. DODD: Vector analysis (first semester), two hours; Non-euclidean geometry (second semester), two hours. The mathematical seminar meets one evening each week.

UNIVERSITY OF MICHIGAN. — By Professor W. W. BEMAN: Solid analytic geometry, two hours; Higher plane curves (second semester), two hours; Differential equations (first semester), three hours; Linear differential equations (second semester), two hours; Quaternions (second semester), two hours; Seminar, two hours. — By Professor A. ZIWET: Projective geometry, three hours; Harmonic analysis, two hours; Advanced mechanics (second semester), three hours; Theory of potential (first semester), three hours. — By Professor J. L. MARKLEY: Theory of functions, three hours; Advanced theory of functions, two hours. — By Professor J. W. GLOVER; Higher algebra, three hours; Theory of annuities and insurance, two hours. — By Dr. A. B. PIERCE; Differential geometry, three hours. — By Mr. E. B. ESCOTT: Theory of numbers, two hours.

UNIVERSITY OF MISSOURI. — By Professor E. R. HEDRICK: Theory of functions, three hours; Advanced calculus, three hours; Higher analysis, three hours. — By Professor L. M. DEFOE: Analytic mechanics, three hours; Fourier's series and potential function, three hours. — By Professor G. A. BLISS: Differential geometry, three hours; Theory of groups,

three hours. — By Dr. L. D. AMES: Infinite series and products, three hours; Galois's theory of substitutions, three hours. — By Mr. L. INGOLD: Theory of equations and determinants, three hours; Elements of projective geometry, three hours; Elements of differential equations, three hours.

UNIVERSITY OF NEBRASKA. — By Professor E. W. DAVIS: Theory of surfaces, two hours; Pure mathematics, two hours. — By Professor CANDY: Differential equations, three hours; Mathematical pedagogy, three hours. — By Professor C. ENGBERG: Theory of probabilities (second semester), three hours; Algebra of quantics, three hours, or Higher plane curves, two hours; Biometry, I, two hours. — By Miss I. SINCLAIR: Geometry of position, three hours, or Calculus of variations, two hours.

UNIVERSITY OF PENNSYLVANIA. — By Professor E. S. CRAWLEY: Higher plane curves, three hours; Solid analytic geometry, two hours. — By Professor G. E. FISHER: Advanced calculus, two hours; Invariants and covariants (first half year), three hours; Linear differential equations (second half year), three hours. — By Professor I. J. SCHWATT: Theory of functions of a real variable, three hours; Infinite series and products, three hours. — By Professor G. H. HALLETT: Theory of groups, three hours; Calculus of variations (first half year), two hours; Lie's theory of continuous groups (second half year), two hours. — By Dr. B. S. EASTON: Algebra (in German), two hours; Theory of higher equations, two hours; Elementary divisors and group characters, two hours. — By Dr. F. H. SAFFORD: Curvilinear coördinates and orthogonal transformations with applications to the theory of potential, three hours.

Summer session (July 5 to August 12, 1905). Each course will be given five hours per week. — By Professor E. S. CRAWLEY: Theory of numbers. — By Professor G. E. FISHER: Invariants and covariants. — By Professor I. J. SCHWATT: Definite integrals. — By Professor G. H. HALLETT: Theory of abstract groups. — By Dr. F. H. SAFFORD: Differential equations.

SYRACUSE UNIVERSITY. — By Professor W. H. METZLER: Analytic geometry (first semester), three hours; Modern geometry (second semester), three hours; Newtonian potential

and spherical harmonics, three hours; General theory of functions, three hours; Determinants, three hours; Elliptic integrals and elliptic functions, three hours. — By Professor E. D. ROE: Theory of invariants and covariants, three hours; Theory of substitutions, three hours; Advanced calculus (first semester), three hours; Differential equations (second semester), three hours; Analytic mechanics, three hours; Theory of equations, two hours; Analytic trigonometry (first semester), one hour; Determinants (second semester), one hour. — By Professor W. G. BULLARD: Projective geometry (first semester), three hours; Higher plane curves (second semester), three hours; Twisted curves and general theory of surfaces, three hours.

UNIVERSITY OF WISCONSIN. — By Professor C. A. VAN VELZER: Differential equations, three hours; Analytic geometry, three hours. — By Professor C. S. SLICHTER: Theoretic mechanics, three hours; Theory of probabilities (second semester), two hours; Hydrodynamics, two hours. — By Professor E. B. SKINNER: Geometry of three dimensions, two hours; Advanced calculus, two hours; Twisted curves and surfaces (first semester), three hours; Quaternions (second semester), three hours: Seminar in groups, two hours. — By ——————: Projective geometry, two hours.

DURING the present summer semester the various foreign universities offer the following courses in mathematics:

UNIVERSITY OF COPENHAGEN (February to June). — By Professor T. N. THIELE: Numerical calculation, four hours. — By Professor H. G. ZEUTHEN: Calculus, six hours; Homogeneous coordinates, one hour; exercises in the history of mathematics, one hour. — By Professor J. PETERSEN: General theory of groups, four hours. — By Professor N. NIELSON: Theory of gamma functions, four hours. — By Dr. C. JUEL: Algebraic and graphic curves, two hours. — By Dr. P. HEEGAARD: Hydrodynamics, two hours.

UNIVERSITY OF CZERNOWITZ. — By Professor D. v. STERNECK: Differential and integral calculus, II, two hours; Theory of functions, three hours; Seminar, two hours.

UNIVERSITY OF GENEVA. — By Professor C. CAILLER: Differential and integral calculus, three hours; Rational mechanics, II, two hours; Seminar, two hours. — By Professor

H. FEHR; Descriptive and projective geometry, two hours; Theory of algebraic equations, two hours; Seminar, two hours. — By Dr. R. DE SAUSSURE: Geometry of motion, two hours. — By Dr. D. MIRIMANOFF: Theory of potential, two hours.

UNIVERSITY OF GRAZ. — By Professor J. FRISCHAUF: Higher analysis with applications to geometry, five hours. — By Professor K. V. v. DANTSCHER: Elements of the theory of functions, and differential calculus, five hours; Seminar, two hours. — By Professor J. STREISSLER: Descriptive geometry, IV, three hours.

UNIVERSITY OF INNSBRUCK. — By Professor O. STOLZ: Theory of real numbers as introduction to the differential calculus, four hours; Theory of functions of a complex variable, one hour; Seminar, two hours. — By Professor K. ZINDLER: Differential equations, three hours; Theory of surfaces, four hours. — By Professor MENGER: Projective geometry, two hours.

GERMAN UNIVERSITY OF PRAGUE. — By Professor G. PICK: Differential and integral calculus, II, four hours; Concept of number and limit, one hour; Seminar, two hours. — By Professor J. GMEINER: Analytic geometry of space, five hours.

UNIVERSITY OF VIENNA. — By Professor G. v. ESCHERICH: Theory of functions, five hours; Theory of probabilities, three hours; Proseminar, one hour; Seminar, one hour. — By Professor F. MERTENS: Algebra, II, five hours: Theory of statistics, three hours; Seminar, two hours; Proseminar, one hour. — By Professor W. WIRTINGER: Differential and integral calculus, five hours; with exercises, two hours; Seminar, two hours; Proseminar, one hour. — By Professor G. KOHN: Synthetic geometry, II, four hours; with exercises, one hour; Non-euclidean geometry, three hours. — By Dr. A. TAUBER: Mathematics of insurance, II, six hours. — By Dr. E. BLASCHKE: Mathematics of statistics, II, three hours. — By Dr. K. CARDA: Linear continuous groups, two hours. — By Dr. J. PLEMELJ: Hypergeometric differential equations, two hours. — By Dr. W. GRÜNWALD: Line geometry, II, two hours.

PROFESSOR F. LINDEMANN, of the University of Munich, has been decorated with the order of St. Michael, of the third class.

PROFESSOR ERNEST LEBON has been elected a corresponding member of the academy of sciences of Lisbon.

PROFESSOR M. DISTELI, of the University of Strassburg, has been appointed professor of descriptive geometry at the technical school at Dresden.

PROFESSOR F. SCHILLING, of the new technical school of Danzig, has declined the call to the technical school at Charlottenburg.

AT the University of Bern, Dr. PEXIDER has been appointed docent in the theory of numbers.

DURING December, 1905, a course of fifteen lectures will be delivered at Columbia University by Professor V. F. BJERKNES, of the University of Stockholm, on "Fields of force," including the discussion of the hydrodynamic analogues of the electrostatic and electromagnetic fields.

A similar course will be delivered in March and April, 1906, by Professor H. A. LORENTZ, of the University of Leyden, on "Extensions of Maxwell's electromagnetic theory of light, and the dynamics of the electron."

These courses are open, without charge, to all teachers and advanced students of physics who may desire to attend. Exact dates and other details may be obtained hereafter on application to the secretary of Columbia University.

DURING Professor E. B. VAN VLECK'S absence in Europe the mathematical department at Wesleyan University will be in charge of Professor J. M. VAN VLECK. Dr. B. C. EUER and Mr. L. A. HOWLAND have been appointed instructors in mathematics.

PROFESSOR J. H. TANNER has been granted leave of absence from Cornell University for next year. Mr. W. H. CARRUTH and Mr. E. C. COLPITTS have been appointed assistants in mathematics.

AT the Massachusetts Institute of Technology, associate professor D. P. BARTLETT has been promoted to a full professorship of mathematics.

Dr. J. V. WESTFALL, instructor in mathematics at the University of Iowa, has been promoted to an assistant professorship at the same institution.

At the University of Indiana, Dr. S. C. Davisson and Dr. D. A. Rothrock have been promoted from associate to junior professorships of mathematics.

Dr. T. W. Wright, professor of applied mathematics at Union College, has resigned on account of ill health.

Professor Leben Warren, for twenty-seven years professor of mathematics at Colby College, Watertown, Maine, died April 21, at the age of 69 years.

Dr. Corneille L. Landré, actuary of the Netherlands general society, died February 10, at the age of sixty-seven. Dr. Landré was a frequent contributor to actuarial and mathematical journals, and was perhaps one of the most eminent of the world's actuaries. His well known work, Mathematisch-technische Kapitel zur Lebensversicherung, easily ranks among the foremost treatises on insurance mathematics.

Catalogues of second-hand mathematical works: Thury, Baumgartner & Co., 4 Rue Diday, Geneva, catalogue No. 53, 148 mathematical titles; Gustav Fock, 7 Schlossgasse, Leipzig, catalogue No. 267, 1862 titles; W. Junk, 22 Rathenowerstrasse, Berlin, catalogue No. 28, 740 titles in mathematics; Theordor Ackermann, 10 Promenadenplatz, Munich, catalogue No. 539, 735 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BAIRE (R.). Théorie des nombres irrationnels, des limites et de la continuité. Paris, Vuibert, 1905. 8vo. 61 pp.

BAUER (D.). Ueber den Teilungskörper der elliptischen Funktionen mit singulärem Modul und die zugehörigen Klassenkörper. (Diss.) Strassburg, 1903. 8vo. 33 pp.

CAMACHO Y SANJURJO (J. J.). Apuntes de geometría analítica. Sevilla, Díaz, 1904. 8vo. 45 pp.

COOLIDGE (J.). Die dual-projektive Geometrie im elliptischen und sphärischen Raume. (Diss.). Bonn, 1904. 8vo. 54 pp.

DANNMEYER (F.). Die Oberflächen- und Volumenberechnung für den Lobatschefskij'schen Raum mit besonderer Berücksichtigung der Rotationskörper und Polyeder. (Diss.) Kiel, 1904. 8vo. 55 pp.

DEDEKIND (R.). Stetigkeit und irrationale Zahlen. 3te unveränderte Auflage. Braunschweig, Vieweg, 1905. 8vo. 7 + 24 pp. M. 1.00

FELÍU Y VEGUÉS (F.). Algunos trabajos matemáticos. Edición publicada por sus discípulos y ofrecida al maestro en testimonio de gratitud. Con un prólogo de S. Mundí y Giró. Barcelona-Gracia, Seix, 1905. 13 + 221 pp.

FIGUEOA MATEU (A.). Cálculo infinitesimal. Primera parte. Vol. I: Nociones generales abstractas de cáclulo diferencial al alcance de los conocimientos elementales de matemáticas. Oviedo, Gusano, 1904. 8vo. 120 pp. Fr. 8.00

GÄSSLER (E.). Ueber senär cyklische Korrelationen in der Ebene und im Raume. (Diss.) Strassburg, 1904. 8vo. 39 pp.

GULDBERG (A.). Ueber lineare Differenzengleichungen. Leipzig, Teubner, 1905. 8vo. (*Verhandlungen des III. Internationalen Mathematiker-Kongresses*, Heidelberg, 1904, pp. 157–163.)

HARTWIG (T.). Schule der Mathematik zum Selbstunterrichte. Neue mathematische Unterrichtsbriefe. (Beilage zu *Das Wissen, für Alle.*) Vol. II: Analytische Geometrie der Ebene und des Raumes. Wien, Perles, 1905. 8vo. 200 pp. M. 2.50

KRAUSE (R.). Ueber senär cyklische Kollineationen im Raume. (Diss.) Strassburg, 1904. 8vo. 57 pp.

MUNDÍ Y GIRÓ (S.). FELÍU Y VEGUES (F.).

NITZ (K.). Anwendungen der Theorie der Fehler in der Ebene auf Konstruktionen mit Zirkel und Lineal. (Diss.) Königsberg, 1905. 8vo. 32 pp.

POMPEIU (D.). Sur la continuité des fonctions de variables complexes. (Thèse.) Paris, Gauthier-Villard, 1905. 4to. 59 pp.

RECKHAUS (H.). Ueber das räumliche Sechseck. (Diss.) Strassburg, 1904. 8vo. 35 pp.

RIEBER (R.). Ueber vier Elemente der Ebene, von denen je drei das vierte und zugleich eine Parabel bestimmen. (Diss.) Strassburg, 1904. 8vo. 44 pp.

SCHMID (E.). Die Cauchy'sche Methode der Auswertung bestimmter Integrale zwischen reellen Grenzen. (Progr.) Cannstadt, 1904. 4to. 93 pp.

SCHRADER (A.). Ueber den Ort der Endpunkte, die man erhält, wenn man auf jeder Tangente eines Kegelschnitts nach beiden Seiten hin vom Berührungspunkte aus eine konstante Strecke abträgt. (Progr.) Paderborn, 1904. 8vo. 47 pp.

SCHULZ-BANNEHR (L.). Zur Invarianten- und Funktionentheorie einer speciellen Curve 4. Ordnung. (Diss.) Strassburg, 1904. 8vo. 50 pp.

VAHLEN (J.). Abstrakte Geometrie. Untersuchungen über die Grundlagen der Euklidischen und Nicht-Euklidischen Geometrie. Leipzig, Teubner, 1905. 8vo. 12 + 302 pp. Cloth. M. 12.00

WALTER (M.). Die gleichseitige Hyperbel. (Diss.) Strassburg, 1904. 8vo. 32 pp.

WEIERSTRASS (K.). O przedstawialności analitycznéj tak zwanych dowolnych funkcyj argumentów rzeczywistych. Warszawa, 1904. 8vo. 36 pp. M. 1.50

WERNICKE (P.). Ueber die analysis situs mehrdimensionaler Räume. (Diss.) Göttingen, 1904. 8vo. 48 pp.

ZORETTI (L.). Sur les fonctions analytiques uniformes qui possèdent un ensemble parfait discontinu de points singuliers. (Thèse.) Paris, Gauthier-Villars, 1905. 4to. 57 pp.

II. ELEMENTARY MATHEMATICS.

BOS (H.). Geometría elemental. Paris, Hachette, 1905. 16mo. 283 pp. Fr. 3.00

CLAUSSEN (F.). Leitfaden der Planimetrie, nebst einer kurzen Anleitung zu trigonometrischen und stereometrischen Berechnungen. 2te, erweiterte Auflage. Leipzig, Hirt, 1905. 8vo. 92 pp. M. 1.00

CRACKNELL (A. G.). See WORKMAN (W. P.).

DESSENON (E.). Traité de trigonométrie rectiligne, à l'usage des élèves de seconde et de première C et D, de mathematiques A et B, et des candidats au baccalauréat et aux écoles. 4e édition. Paris, Vuibert, 1905. 8vo. 2 + 304 pp.

DOMINGUEZ BERRUETA (J.). Trigonometría. Para uso de los alumnos del bachillerato conforme á los últimos métodos de enseñanza, Salamanca, 1904. 8vo. 60 pp.

ELÉMENTS de géométrie, comprenant des notions sur les courbes usuelles, un complément sur le déplacement des figures et de nombreux exercices. Paris, Poussielgue, 1905. 16mo. 12 + 525 pp.

EYSSÉRIC ET PASCAL. Eléments de géométrie et notions sur les courbes usuellés, à l'usage des élèves de l'enseignement secondaire et des candidats aux baccalauréats. 23e édition, revue et augmentée, conforme aux programmes du 31 mai 1902. Paris, Delagrave, 1904. 12mo. 526 pp.

FENKNER (H.). Arithmetische Aufgaben. Unter besonderer Berücksichtigung von Anwendungen aus dem Gebiete der Geometrie, Physik und Chemie. Für den mathematischen Unterricht an höheren Lehranstalten bearbeitet, Ausgabe A. Vornehmlich für den Gebrauch in Gymnasien, Realgymnasien und Ober-Realschulen. Teil IIa: Pensum der Obersekunda. 3te, vermehrte Auflage. Berlin, Salle, 1905. 8vo. 3 + 114 pp. M. 1.20

FOURREY. Cours d'algèbre. (Cours moyens.) 4e édition. Paris, 1905. 8vo. 259 pp.

GRANDPIERRE (A.). Livret de géométrie. (Cours élémentaire et moyen.) Rouen, 1904. 12mo. 22 pp.

GRÉVY (A.). Eléments d'algèbre, à l'usage des élèves des classes de troisième A à première A et B (programmes du 31 mai 1902). 5e édition. Paris, Vuibert, 1905. 16mo. 210 pp. Fr. 1.75

——. Eléments de géométrie à l'usage des élèves des classes de seconde et première A et B (programmes du 31 mai 1902). 5e édition. Paris, Vuibert, 1905. 16mo. 136 pp. Fr. 1.25

——. Géométrie (compléments), à l'usage des élèves des classes de seconde et première D (programmes du 31 mai 1902). 3e édition. Paris, Vuibert, 1904. 16mo. 120 pp.

——. Algèbre, à l'usage des élèves des classes de quatrième B à première C et D (programmes du 31 mai 1902). 3e édition. Paris, Vuibert, 1905. 16mo. 350 pp. Fr. 2.50

——. Eléments de géométrie, à l'usage des élèves des classes de quatrième et troisième A (programmes du 31 mai 1902). 5e edition. Paris, Vuibert, 1905. 16mo. 172 pp. Fr. 1.50

——. Géométrie, à l'usage des élèves des classes de cinquième B à troisième B (programmes du 31 mai 1902). 3e édition. Paris, Vuibert, 1905. 16mo. 223 pp. Fr. 2.25

——. Traité d'algèbre, à l'usage des élèves de mathématiques A et B (programmes du 31 mai 1902). Paris, Vuibert, 1905. 8vo. 502 pp. Fr. 6.00

HAACKE (F.). Entwurf eines arithmetischen Lehrganges für höhere Schulen. Leipzig, Teubner, 1904. 8vo. 53 pp. M. 0.80

MANDOUCE (E.). Méthode nouvelle de calculs abrégés. Bordeaux, Maleyre, 1905. 8vo. 83 pp.

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